Accuracy of Homogeneous Sliding Modes in the Presence of Fast Actuators

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Abstract— It is shown in the paper that the higher the order r of the homogeneous sliding mode the less sensitive it is to the presence of unaccounted-for fast stable actuators. In particular, the advantage with respect to standard sliding modes is revealed. With the actuator time constant $\mu << 1$ the sliding variable magnitude is proved to be proportional to μ^r.

Key words: High-order sliding mode; Robustness; Homogeneity; Chattering; Output feedback.

I. Introduction

Sliding mode control is one of the main control tools under heavy uncertainty conditions. It is accurate and insensitive to disturbances [4, 6, 31, 32, 33]. The main drawback of the standard sliding modes is mostly related to the so-called chattering effect (dangerous high-frequency system vibrations) caused by high, theoretically infinite control switching frequency, and much research was devoted to avoiding it [1, 5, 10, 11, 12, 14, 26, 29, 30, 32-34].

High order sliding modes (HOSMs) [2, 3, 6, 8, 17-19, 26-28] were created to remove the restrictions of standard sliding modes hiding the switching in the higher derivatives of the system coordinates. Their practical application requires the robustness to be shown with respect to various possible imperfections. Performance of the 2-sliding sub-optimal controller was recently studied in the presence of time delays [16]. The robustness of general homogeneous sliding modes was proved by Levant [19] with respect to various switching imperfections, small delays and noises. In all cases the sliding variable vibrations are shown to be small.

In reality the control does not directly influence the system, but enters a special device called actuator, being itself a dynamic system. The purpose of the actuator is to properly transmit the input, and it performs well, when the input changes smoothly and slowly. For this end the actuator is to be fast, exact and stable. Presence of such actuators may however destroy the sliding mode. Indeed, a high frequency discontinuous input causes uncontrolled vibrations of the actuator and of its output. In particular, this causes vibrations of the 2-sliding control systems [5, 10-12].

The robustness of homogeneous HOSMs with respect to the presence of fast actuators is for the first time shown in this paper. The corresponding asymptotic sliding accuracy is estimated with respect to the small actuator time constant. The accuracy order appears to be determined by the sliding-mode order only. Thus, the higher the sliding-mode order the less sensitive is the HOSM to fast actuators. The homogeneity requirement is satisfied for almost all known HOSM controllers. Simulation confirms the theoretical results.

II. The problem statement

Consider a smooth dynamic system with a smooth output function σ. Let the system be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense [9]. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables; and the set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is called r-sliding (rth order sliding) mode [17,18]. The standard sliding mode, used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system

$$\dot{x} = a(t,x) + b(t,x)v, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, the scalar system output $\sigma(t, x)$ is measured in real time, $v \in \mathbf{R}$ is the input, n is uncertain. The task is to provide in finite time for keeping σ close to zero using sampled values of σ.

Assumption 1°. Smooth uncertain functions a, b and σ are defined in some region $\Omega \subset \mathbf{R}^{n+1}$. It is supposed that provided the input v is a Lebesgue-measurable function of time, $|v| \leq v_M$, all solutions starting from an open region $\Omega_x \subset \mathbf{R}^n$ at $t \in t_a$ can be extended in time up to $t = t_b > t_a$ without leaving the region Ω. The constant $v_M > 0$ is defined in Assumption 4°.

Note that actually a weaker assumption is needed that any *feasible* trajectory does not leave Ω. Not all inputs can be realized.

Assumption 2º. The relative degree r of the system is assumed to be constant and known. That means that for the first time the input variable v appears explicitly in the rth total time derivative of σ [15]. It can be checked [15] that

$$\sigma^{(r)} = h(t,x) + g(t,x)v, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{v=0}$, $g(t,x) = \frac{\partial}{\partial v}\sigma^{(r)}$ are some unknown smooth functions, which can be expressed in the terms of Lie derivatives. The set Ω_x at the time t_a is supposed to contain r-sliding points, i.e. points satisfying $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$.

The local consideration is natural here, since the asymptotic accuracy is searched for. Note that in the standard problem statement $\Omega_x = \mathbf{R}^n$, $t_b = \infty$, and the results are global [18,19,25].

Assumption 3º. It is supposed that

$$0 < K_m \leq \frac{\partial}{\partial v}\sigma^{(r)} \leq K_M, \qquad |\, \sigma^{(r)}|_{v=0} \,| \leq C \tag{3}$$

hold in Ω for some K_m, K_M, $C > 0$. Conditions (3) are formulated in terms of input-output relations.

The actuator is described by the equations

$$\mu \dot{z} = f(z, u), \qquad v = v(z) \tag{4}$$

where $z \in \mathbf{R}^m$, $u \in \mathbf{R}$ is the control and the input of the actuator, v is a continuous output function, the time constant $\mu > 0$ is a small parameter.

The control u is determined by a feedback

$$u = U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \tag{5}$$

where U is a function continuous almost everywhere, and bounded by some constant u_M, $u_M > 0$, in its absolute value. Being applied directly to (1), i.e. with

$$v = u, \tag{6}$$

it locally establishes the r-sliding mode $\sigma \equiv 0$. All differential equations are understood in the Filippov sense [9]. In order to apply (5) one needs to estimate σ and its derivatives.

Assumption 4º. The actuator features Bounded-Input-Bounded-State (BIBS) property with $\mu = 1$. Initial values of z belong to some compact set. Since $|u| \leq u_M$ this provides for infinite extension in time of any solution of (4) and z belonging to some compact region Ω_z independent of μ. Indeed, μ can be excluded by the time transformation $\tau = t/\mu$. This assumption causes also the actuator output v to be bounded in its absolute value by some constant $v_M > u_M > 0$.

Assumption 5º. The dynamic output-feedback (5) is supposed to be r-sliding homogeneous [19], which means that the identity

$$U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \equiv U(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) \quad (7)$$

is kept for any $\kappa > 0$. Most known HOSM controllers [2, 17-21, 25] satisfy this assumption. It is also assumed that the control function U is locally Lipschitzian everywhere except a finite number of smooth manifolds comprising a closed set Γ in the space with coordinates $\Sigma = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$. Note that due to the homogeneity property (7) the set Γ contains the origin $\Sigma = 0$, where the function U is inevitably discontinuous [19,20].

As follows from (2), (3)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, v\,. \quad (8)$$

This inclusion does not "remember" anything on system (1) except the constants r, C, K_m, K_M.

Assumption 6º. It is assumed that with control (5) applied directly to inclusion (8) a finite-time stable inclusion (5), (6), (8) is created. This is the standard way to implement HOSM controllers [19].

The differential inclusions are understood here in the Filippov sense. That means that the right-hand vector set is enlarged in a special way [19] at the discontinuity points of U in order to satisfy certain convexity and semicontinuity conditions [9].

Assumption 7º. The actuator is assumed exact in the following sense. With $\mu = 1$ and any constant value of u the output v uniformly tends to u. That means that for any $\delta > 0$ there exists $T > 0$ such that with any u, u = const, $|u| \le u_M$, $z(0) \in \Omega_z$, the inequality $|v - u| \le \delta$ is kept after the transient time T. It is required also that the function $f(z, u)$ in (4) be uniformly continuous in u, which means that $\|f(z, u) - f(z, u + \Delta u)\|$ tends to 0 with $\Delta u \to 0$ uniformly in $z \in \Omega_z$, $|u| \le u_M$.

Note that any linear actuator with the transfer function $P(\mu p)/Q(\mu p)$, where deg Q – deg $P > 0$, Q is a Hurwitz polynomial, $P(0)/Q(0) = 1$, satisfies Assumptions 4º, 7º. While Assumptions 1°-7° can be considered natural, the next technical Assumption is to be separately checked for each controller (5).

Assumption 8º. It is supposed that the change of (5), (6) at the set Γ to

$$v \in \begin{cases} U(\Sigma), & \Sigma \notin \Gamma \\ [-v_M, v_M], & \Sigma \in \Gamma \end{cases} \tag{9}$$

does not interfere with the finite-time convergence, i.e. (8), (9) is finite-time stable. Recall that $\Sigma = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$.

Note that while Filippov solutions of discontinuous differential equations do not depend on the values of the right-hand side on any set of the measure 0, it is not true with respect to differential inclusions. Since $v_M > u_M$, solutions of (8), (9) contain all solutions of (5), (6), (8). Note also that the inclusion (8), (9) is also r-sliding homogeneous, and, therefore, its asymptotic stability is equivalent to the finite-time stability [19,26].

III. Main results

Theorem 1. *Let assumptions* 1º - 8º *hold. Then there exist a vicinity Q of the r-sliding set in* Ω_x *at* $t = t_a$, $t_1 \in (t_a, t_b)$ *and* $a_0, a_1, ..., a_{r-1} > 0$, *such that with sufficiently small* μ *for any trajectory of* (1), (4), (5) *starting within Q at* $t = t_a$ *the inequalities* $|\sigma| < a_0\mu^r$, $|\dot{\sigma}| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ *are kept with* $t \ge t_1$.

Note that a global Theorem holds in the global case [19] with $t_b = \infty$. Here and further the proofs are presented in Section IV. Assumption 8º is to be checked for each controller. Fortunately it holds for most known HOSM controllers. Consider some of them.

Three known families of high-order sliding controllers are considered here

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \qquad (10)$$

which are defined by recursive procedures. In the following $\alpha, \beta_1,..., \beta_{r-1} > 0$ and $i = 1,..., r-1$.

1. The following procedure defines the "nested" r-sliding controller (Levant 2003), based on a pseudo-nested structure of 1-sliding modes. Let q be the least common multiple of 1, 2, ..., r. Controller (10) is built as follows:

$$N_{i,r} = (|\sigma|^{q/r} + |\dot{\sigma}|^{q/(r-1)} + ... + |\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q}; \quad \Psi_{0,r} = \text{sign}\, \sigma, \quad \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}).$$

2. Controller (5) is called quasi-continuous (Levant, 2005b), if it can be redefined according to continuity everywhere except the r-sliding set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$. The following procedure defines a family of quasi-continuous controllers (Levant, 2005b):

$$\varphi_{0,r} = \sigma, \ N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r} / N_{0,r} = \text{sign}\, \sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)} \Psi_{i-1,r}, \quad N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)}, \quad \Psi_{i,r} = \varphi_{i,r} / N_{i,r}.$$

3. Let $\text{sat}(z, \varepsilon) = \min[1, \max(-1, z/\varepsilon)]$. Another family of quasi-continuous controllers (Levant, 2005a) is obtained from the first family by the following homogeneous regularization:

$$N_r = (|\sigma|^{q/r} + |\dot{\sigma}|^{q/(r-1)} + ... + |\sigma^{(r-1)}|^{q})^{1/q}, \quad \psi_{0,r} = \text{sign}\, \sigma, \ \psi_{i,r} = \text{sat}([\sigma^{(i)} + \beta_i N_{i,r} \psi_{i-1,r}] / N_r^{r-i}, \varepsilon_i).$$

Following are the nested sliding-mode controllers (of the first family) for $r \le 4$ with tested β_i and q being the least multiple of 1,..., r:

1. $u = -\alpha\, \text{sign}\, \sigma$,
2. $u = -\alpha\, \text{sign}(\dot{\sigma} + |\sigma|^{1/2} \text{sign}\, \sigma)$,
3. $u = -\alpha\, \text{sign}(\ddot{\sigma} + 2\,(|\dot{\sigma}|^3 + |\sigma|^2)^{1/6}\ \text{sign}(\dot{\sigma} + |\sigma|^{2/3} \text{sign}\, \sigma))$,
4. $u = -\alpha\, \text{sign}\{\dddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^3)^{1/12} \text{sign}[\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{1/6} \text{sign}(\dot{\sigma} + 0.5\, |\sigma|^{3/4} \text{sign}\, \sigma)]\}$.

It can be shown that the above sets of parameters β_i with $r \le 4$ are valid for all 3 families of controllers, $\varepsilon_i = 0.2$ is chosen in that case for the third family. Note that while enlarging α increases the class (3) of systems, to which the controller is applicable, parameters β_i, ε_i are tuned to provide for the needed convergence rate. The procedure of choosing the parameters is proposed in [24].

Theorem 2. *The listed* 3 *families of arbitrary-order sliding-mode controllers satisfy Assumption* 8º.

Also generalized controllers [25] as well as all 2-sliding controllers from [17,21], which do not require switching on the axis $\dot{\sigma} = 0$, can be shown to satisfy Assumption 8º. The popular sub-optimal 2-sliding controller [2,3,8] does satisfy Assumption 8º, though it has memory and, therefore, does not have the form (5). As follows from the proof, Theorem 1 is true also for it.

Unfortunately, the twisting controller

$$u = -\alpha_1 \operatorname{sign} \sigma - \alpha_2 \operatorname{sign} \dot{\sigma}, \qquad \alpha_1 > \alpha_2 > 0 \tag{11}$$

does not satisfy Assumption 8º. Indeed, the set Γ consists of the axes $\sigma = 0$, $\dot{\sigma} = 0$, and (10) having been applied, the possibility of the sliding mode $\dot{\sigma} = 0$ appears, preventing the convergence of σ to 0. Nevertheless, the switching logic can be changed preserving the same trajectories, if the twisting controller (11) is considered as a particular case of the generalized sub-optimal controller with the switching parameter $\beta = 0$ [2]. Another way is to require the following assumption.

Assumption 9º. There is a constant $k > 0$ such that for any $\varepsilon > 0$ with sufficiently small $\delta > 0$ the reaction of the actuator output $v(t)$ to a step-wise change Δu of any constant input u at the moment t_0 with $|v(t_0) - u| < \delta$ satisfies the inequality $|v(t) - u| < \varepsilon + k\,|\Delta u|$ for any $t > t_0$.

In the case of a linear actuator this assumption is satisfied if the overshoot of the reaction to a step function does not exceed 100 k %.

As follows from the Assumptions 1º, 2º, the rth derivative of the output σ is uniformly bounded by $C + K_M v_M$. Thus, an r-th order exact robust homogeneous differentiator [18] with finite-time convergence can be applied here, producing exact estimations of $\dot{\sigma}, \ldots, \sigma^{(r-1)}$ and due to the homogeneity preserving the asymptotics from Theorem 1 [19]. It can be shown that the resulting system is robust with respect to small measurement noises.

Remark 1. In practice, one cannot expect the output of the actuator to perform in the only possible way to prevent the convergence of the twisting controller by establishing sliding motion on the axis $\dot{\sigma} = 0$. Therefore, Assumption 7° is probably redundant.

Remark 2. A slightly generalized Assumption 7° can be considered, when the actuator instead of tracking the input u tracks γu, where $\gamma > 0$ is some uncertain constant. The listed HOSM controllers are capable to compensate for such a systematic actuator error, if their output is proportionally increased.

III. Proofs

Lemma 1. *Under assumptions* 1° - 6° *suppose that for some* $\mu = \mu_0$ *there exist a ball* B *centered at* $\Sigma = 0$ *and a bounded invariant set* Θ *in finite time attracting all trajectories of the inclusion* (4), (5), (8) *starting from* $B\times\Omega_{z0}$. *Then the statement of Theorem* 1 *holds.*

Proof of Lemma 1. Due to the homogeneity property (7) with $\kappa > 0$ the transformation

$$G_\kappa: (t, \Sigma, z, \mu) \mapsto (\kappa t, d_\kappa \Sigma, z, \kappa\mu),\;\; d_\kappa: (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \mapsto (\kappa^r \sigma, \kappa^{(r-1)} \dot{\sigma}, ..., \kappa \sigma^{(r-1)}), \qquad (11)$$

transfers the trajectories of the inclusion (4), (5), (8) into the trajectories of the same inclusion but with the actuator parameter changed from μ to $\kappa\mu$. Choose any t_2, $0 < t_2 < t_b - t_a$. Let $\Theta \subset \Theta_\Sigma \times \Theta_z$, where $\Theta_\Sigma \subset \mathbf{R}^r$, $\Theta_z \subset \Omega_z$ are some bounded regions. With λ small enough get that with $\mu = \lambda\mu_0$ the trajectories starting from $d_\lambda B\times\Omega_{z0}$ converge to the invariant set $\Theta_1 \subset d_\lambda\Theta_\Sigma \times \Theta_z$ in the time t_2.

Let the set $d_\lambda\Theta_\Sigma \times \Theta_z$ satisfy the inequalities $|\sigma^{(i)}| \leq \gamma$, $i = 0,..., r$-1, and $||z|| \leq \gamma$, then for an arbitrary parameter μ and $\kappa = \mu/(\lambda\mu_0)$ obtain that (11) transfers $\Theta_1 \subset d_\lambda\Theta_\Sigma \times \Theta_z$ into $\Theta_2 \subset d_{\kappa\lambda}\Theta \times \Theta_z$ being the invariant set of (4), (5), (8). The set $d_{\kappa\lambda}\Theta$ satisfies the inequalities $|\sigma| < a_0\mu^r$, $|\dot{\sigma}| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ with $a_i = \gamma(\lambda\mu_0)^{i-r}$. The new convergence time does not exceed $\mu\, t_2/(\lambda\mu_0)$.

Define $Q \subset \Omega_x$ as the subset of points with Σ belonging to $d_\lambda B$ at $t = t_a$, and let $t_1 = t_a + t_2$.■

Lemma 2. *Under Assumptions* 2°, 5° *let the input* $u(t)$ *of the actuator* (4) *be a Lipschitz function of time* $u(t)$ *with some fixed Lipschitz constant. Then for any* $\delta, \varepsilon > 0$ *with sufficiently small* μ *the inequality* $|v - u| \leq \varepsilon$ *is established in the time* δ *and is kept afterwards.*

Proof. Let the Lipschitz constant of $u(t)$ be $L > 0$. Consider the time transformation $t = \mu\,\tau$. Then (4) takes the form

$$\dot{z} = f(z, u_1(\tau)), \qquad v = v(z), \quad u_1(\tau) = u(\mu\,\tau).$$

The function $u_1(\tau)$ is also Lipschitzian, but with the Lipschitz constant μL. Fix some initial value t_0 of the time t corresponding to $\tau = t_0/\mu$. Let $T > 0$ be the time τ needed to establish the inequality $|v - u_1| \leq \varepsilon/4$ with any constant $u_1 = u_0$, $|u_0| \leq u_M$. Take $u_0 = u_1(t_0/\mu) = u(t_0)$. With sufficiently small μ the change of u_1 is arbitrarily small during the time 2T. Thus, since the function f is uniformly continuous in the actuator input, and due to the continuous dependence of the solution $z(t)$ on the right-hand side, the inequality $|v - u_1| \leq \varepsilon/2$ is established in time T, and $|v - u_1| \leq \varepsilon$ is kept during the next interval of the same length. Applying the same reasoning from the moment $\tau = t_0/\mu + T$ obtain taking $u_0 = u_1(t_0/\mu + T) = u(t_0 + \mu T)$ that $|v - u_1| \leq \varepsilon$ holds also during the third interval. Continuing this reasoning, obtain that $|v - u_1| \leq \varepsilon$ is kept forever. Returning to the original time $t = \mu\,\tau$ obtain the statement of the Lemma. ■

Proof of Theorem 1. Consider some closed vicinity of the origin $\Sigma = 0$. Let t^* be the maximal time of convergence to 0 for the trajectories of (8), (9) starting in this vicinity. The set Θ of points of the trajectory segments of the length t^* starting in the chosen vicinity of $\Sigma = 0$ is a compact region [9], which is obviously invariant with respect to (8), (9). Moreover, due to the finite-time stability of the inclusion it attracts any trajectory of (8), (9) in finite time. The same is naturaly true with respect to (5), (6), (8), since its solutions satisfy also (8), (9). Note that, due to the homogeneity, any set $d_\kappa\Theta$, $\kappa > 0$, features the same properties and $d_\eta\Theta \subset d_\kappa\Theta$ with $\kappa > \eta > 0$, where d_κ is defined in (12).

Denote $d_{1+}p = \{d_\kappa\, p \mid \kappa \geq 1\}$, $d_+p = \{d_\kappa\, p \mid \kappa \geq 0\}$, and let $O_\delta(p)$ be the closed δ-vicinity of the point $p \in \mathbf{R}^r$, i.e. the points distanced from p by not more than δ. Obviously $d_{1+}\Gamma = d_+\Gamma = \Gamma$, $O_\delta(0) \subset. O_\delta(\Gamma)$, since $0 \in \Gamma$, and $d_{1+}O_\delta(0) = \mathbf{R}^r$.

Take some small $1 > \delta > 0$. According to the definition of the control-singularity set Γ, the function U has a Lipschitz constant L valid in the whole precompact set $\Theta \setminus O_{\delta/2}(\Gamma)$. Respectively, the functions $U(\Sigma(t))$ calculated along the trajectories of the inclusion $\sigma^{(r)} \in [-v_M, v_M]$ in $\Theta \setminus O_{\delta/2}(\Gamma)$ have a common Lipschitz constant L. Then the functions $U(\Sigma(t))$ have the same Lipschitz constant in the whole set $\mathbf{R}^r \setminus H_{\delta/2}(\Gamma)$, where $H_{\delta/2}(\Gamma) = d_{1+}[\Theta \cap O_{\delta/2}(\Gamma) \setminus O_{\delta/2}(0)] \cup O_{\delta/2}(0)$. Indeed, any trajectory $\Sigma(t)$ lying in $\mathbf{R}^r \setminus H_{\delta/2}(\Gamma)$ and passing through $\Sigma(t_0)$ can be locally represented as $d_\kappa \Sigma_1(\kappa(t - t_0))$, where $\Sigma_1(t - t_0) \in \Theta \setminus O_{\delta/2}(\Gamma)$, $\Sigma(t_0) = d_\kappa \Sigma_1(0)$, $\kappa \geq 1$. Thus, $U(\Sigma(t)) = U(d_\kappa \Sigma_1(\kappa^{-1}(t - t_0))) = U(\Sigma_1(\kappa^{-1}(t - t_0)))$, and its Lipschitz constant is $L/\kappa \leq L$.

It follows now from Lemma 2 that with sufficiently small μ after arbitrarily short transient, whose length is determined by μ, the trajectories of (4), (5), (8) satisfy (8) and

$$v \in \begin{cases} U(\Sigma) + [-\gamma, \gamma], & \Sigma \notin H_\delta(\Gamma) \\ [-v_M, v_M], & \Sigma \in H_\delta(\Gamma) \end{cases}, \tag{13}$$

where, as always, the inclusion (8), (13) is understood in the Filippov sense, i.e. is replaced by the minimal Filippov inclusion [19]. Indeed, $v \in [v_M, v_M]$ is always true; the velocity $\dot{\Sigma}$ is uniformly bounded; and the minimal distance between $H_{\delta/2}(\Gamma)$ and $\mathbf{R}^r \setminus H_\delta(\Gamma)$ is actually determined inside the region Θ, since d_κ enlarges distances with $\kappa > 1$. Thus, the minimal time needed to reach $\mathbf{R}^r \setminus H_\delta(\Gamma)$ from $H_{\delta/2}(\Gamma)$ is separated from zero by some $\tau_m > 0$. Now, starting from the initial time plus τ_m, any point of the trajectory of (4), (5), (8) in $\mathbf{R}^r \setminus H_\delta(\Gamma)$ is preceded by a trajectory segment of the time length τ_m which lies in $\mathbf{R}^r \setminus H_{\delta/2}(\Gamma)$. According to Lemma 2, it is sufficient now to take μ so small that the transient time, corresponding to the Lipschitz constant L, be less than τ_m, providing for $|v - u| \leq \gamma$ thereafter.

Thus, differential inclusion (8), (13) is to be considered now, which in its turn is a perturbation of (8) and

$$v \in \begin{cases} U(\Sigma)+[-\gamma,\gamma], \ \Sigma \notin d_+(O_\delta(\Gamma)\setminus O_\delta(0)) \\ [-v_M, v_M], \ \Sigma \in d_+(O_\delta(\Gamma)\setminus O_\delta(0)) \end{cases}. \qquad (14)$$

Differential inclusion (8), (14) can be considered as a small homogeneous perturbation of the r-sliding homogeneous finite-time-stable differential inclusion (8), (9). Therefore, (8), (14) is also finite-time stable with sufficiently small γ and δ [19]. The Theorem follows now from Lemma 1, since (13) and (14) differ only in the vicinity $O_\delta(0)$ of the origin.■

Proof of Theorem 2. The change of (5), (6), (8) to (8), (9) can generate new motions only on the singularity set Γ. Supposing that the trajectory immediately leaves Γ obtain a trajectory which satisfies (5), (6), (8) almost everywhere, i.e. one of the solutions of (5), (6), (8). Thus, any new motion is to remain on Γ during some time interval. It means that such motion can appear only in the points $\Sigma \in \Gamma$, where the vector set $(\dot\sigma, ..., \sigma^{(r-1)}, [-v_M, v_M])^t$ contains a vector tangential to Γ. Call it the tangentiality condition.

In the case of the nested r-sliding controller (*the first family*) Γ consists of the discontinuity set of the control and of the points where $N_{i,r} = 0$, $i = 1,..., r - 1$, and, therefore, the control is not Lipschitzian. The set $N_{i,r} = 0$ is the set $\sigma = \dot\sigma = ... = \sigma^{(i-1)} = 0$, and the above tangentiality condition requires $\Sigma \equiv 0$. Such a motion satisfies also (5), (6), (8) and is not a new one.

Consider now the discontinuity set. The main point of the convergence proof for the first family of controllers is that after some transient the trajectory never leaves some relatively small vicinity of the discontinuity set determined by the control gain α [18]. Any new motions on the discontinuity set do not interfere with this proof, which establishes the Lemma for the first family.

In the case of *the second family* of controllers Γ contains only points where $N_{i,r} = 0$, $i = 1,..., r – 1$, with differently defined $N_{i,r}$ (see Section III). It can be shown by induction that $N_{i,r} = 0$ iff $\sigma = \dot\sigma = ... = \sigma^{(i)} = 0$. Thus, the tangentiality condition requires $\Sigma \equiv 0$. This is not a new motion. Thus, in this case solutions of (5), (6), (8) and (8), (9) coincide.

The singularity set Γ of *the third family* consists only of the origin $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$, which makes the Theorem trivial in that case. ■

IV. Simulation

Already traditional example of the kinematic car model

$$\dot{x} = v \cos \varphi, \qquad \dot{y} = v \sin \varphi, \qquad \dot{\varphi} = \tfrac{v}{l} \tan \theta, \quad \dot{\theta} = u_{act},$$

is chosen. Here x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1a), u_{act} is the actuator output. The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while $g(x)$ and y are assumed to be measured in real time. Let $v = const = 10$ m/s, $l = 5$ m, $g(x) = 10 \sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$.

Define $\sigma = y - g(x)$. The relative degree of the system is 3 and 3-sliding controller can be applied here. A representative of the less known third family was chosen for demonstration. The resulting output-feedback controller (7), (8) is defined as

$$N_3 = (|w_0|^2 + |w_1|^3 + |w_2|^6)^{1/6}, \qquad \text{sat}(p,0.2) = \min[1, \max(-1, 5p)],$$

$$u = - 0.5\ \text{sat}\{[w_2 + 2(|w_1|^3 + |w_0|^2)^{1/6} \text{sat}((w_1 + |w_0|^{2/3} \text{sign}\ \sigma) / N_3, 0.2)] / N_3,\ 0.2\},$$

where w_i are the real time estimations of the derivatives $\sigma^{(i)}$, $i = 0, 1, 2$, obtained by the differentiator

$$\dot{w}_0 = \xi_0, \qquad \xi_0 = - 9\ | w_0 - \sigma|^{2/3} \text{sign}(w_0 - \sigma) + w_1,$$

$$\dot{w}_1 = \xi_1, \quad \xi_1 = - 15\ | w_1 - \xi_0|^{1/2} \text{sign}(w_1 - \xi_0) + w_2,$$

$$\dot{w}_2 = - 110\ \text{sign}(w_2 - \xi_1).$$

The initial conditions of the differentiator are $w_0(0) = \sigma(0)$, $w_1(0) = w_2(0) = 0$.

The control is applied only starting from $t = 1$ in order to provide some time for the differentiator convergence. The actuator is described by the transfer function

$$F(s) = \frac{\mu s + 1}{\mu^3 s^3 + 2\mu^2 s^2 + 2\mu s + 1}$$

realized in the form

$$\mu \dot{z}_1 = z_2, \quad \mu \dot{z}_2 = z_3, \quad \mu \dot{z}_3 = - z_1 - 2 z_2 - 2 z_3 + u, \qquad u_{act} = z_1 + z_2,$$

with zero initial conditions.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $t = 10^{-4}$. Tracking accuracies are listed in Table 1. It is seen that the accuracies of Σ, $\dot{\sigma}$, $\ddot{\sigma}$ are proportional to μ^3, μ^2, and μ respectively (Fig. 1d, e, f). It is seen from Fig. 1c that the differentiator convergence takes about 0.9 s. The system performs remarkably well with a rather large actuator time constant μ = 0.08. Indeed, the tracking deviation is only 4 cm. (Fig. 1b).

TABLE I: TRACKING ACCURACIES WITH DIFFERENT ACTUATOR TIME CONSTANTS

μ	Sup |σ|	Sup $\|\dot{\sigma}\|$	Sup $\|\ddot{\sigma}\|$
0.01	0.0000765	0.00294	0.189
0.02	0.000644	0.0102	0.374
0.04	0.00529	0.0408	0.746
0.08	0.0433	0.182	1.50

The actuator performance and the resulting steering angles are demonstrated in Fig. 2. Since sliding control signals are actually very fast, one can see from Fig. 2b, d that the actuator performs as a low-pass filter. On the first glance this contradicts the idea of the proofs that the actuator tracks the input with good precision, if the coordinates are distanced from the 3-sliding manifold. In fact, such tracking would be observed here only with very small μ, which in its turn also requires a very small integration step.

Actuators with other transfer functions were also considered providing for similar simulation results.

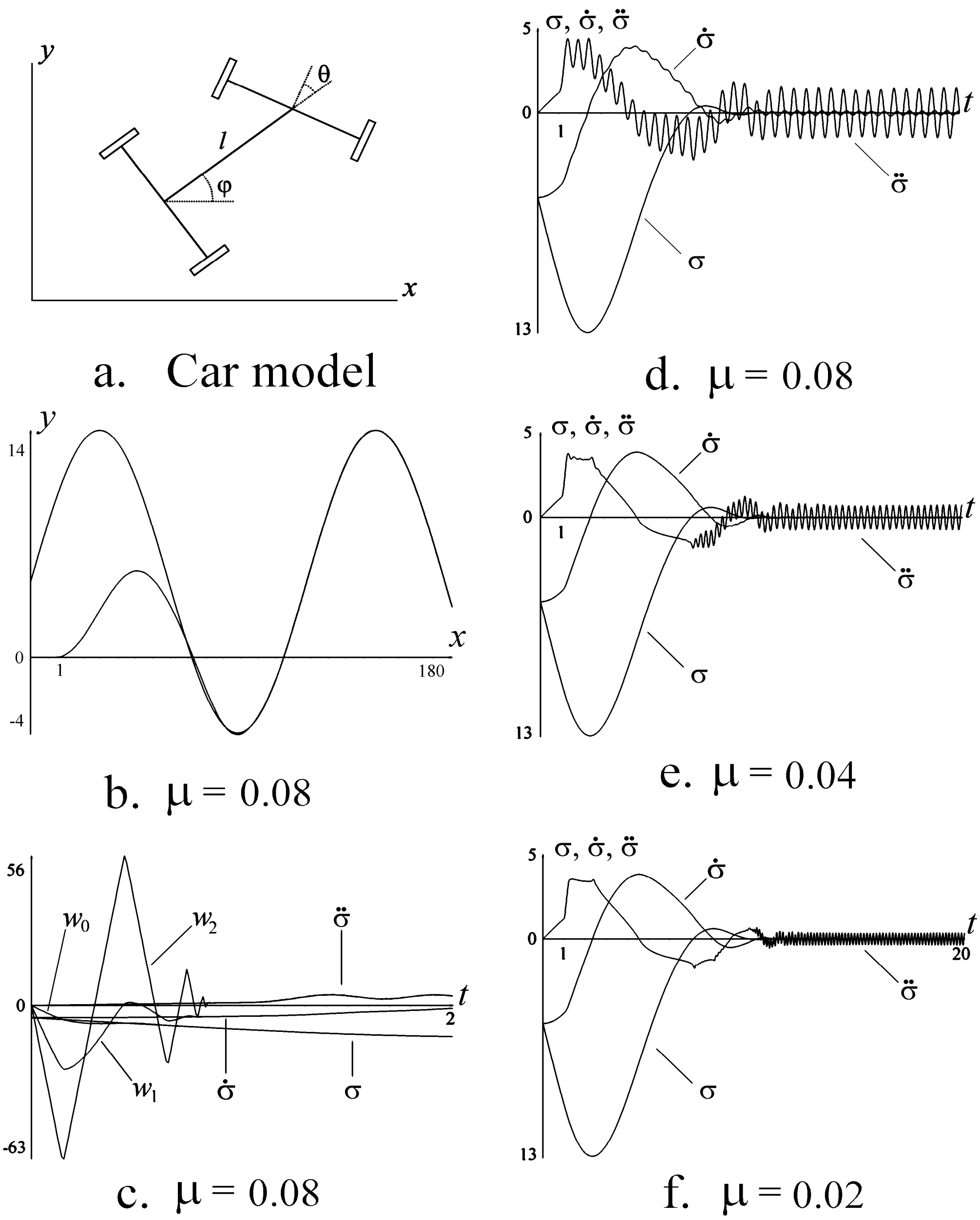

Fig. 1: Car model (a), trajectory tracking (b) and differentiator convergence (c) with μ = 0.08; comparison of 3-sliding deviations with μ = 0.08, 0.04, 0.02 (d, e, f)

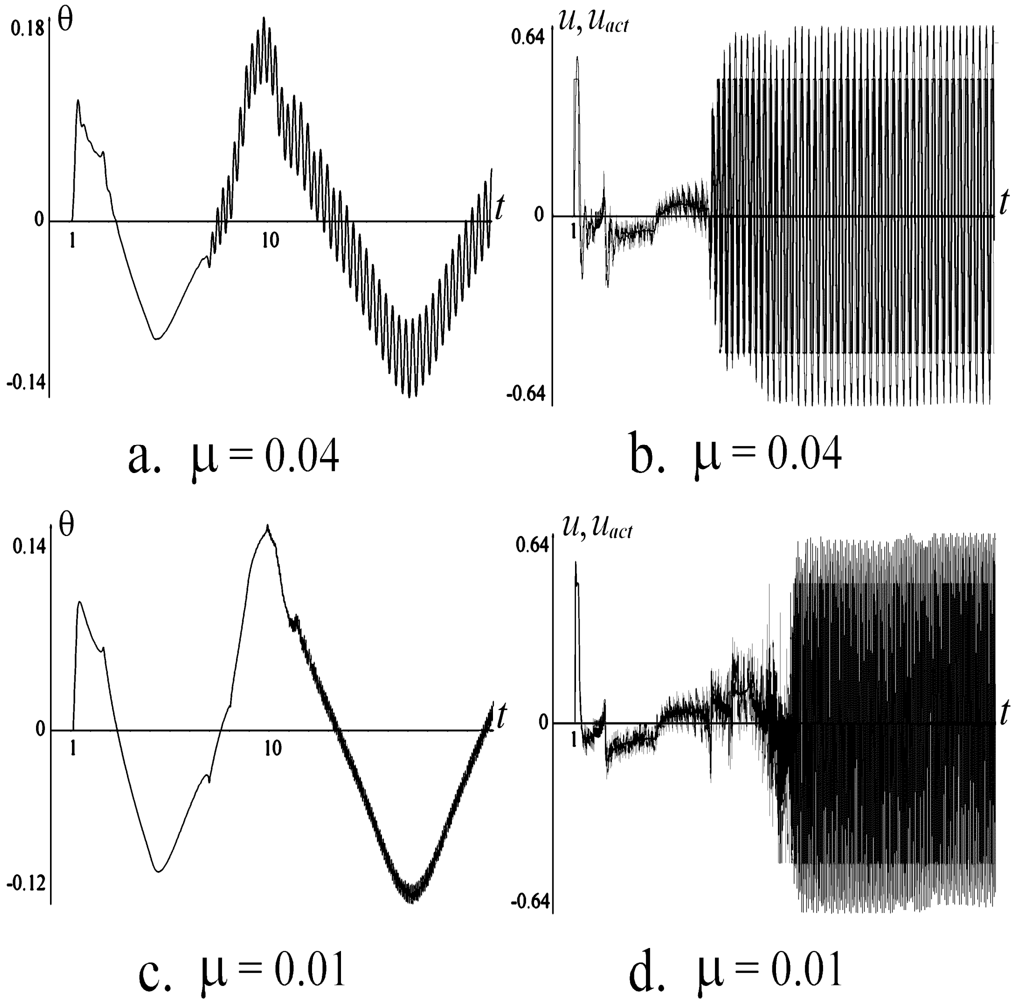

Fig. 2: Steering angle (a, c) and actuator performance (b, d) with $\mu =$ 0.04, 0.02

V. Conclusions

The main conclusion is that stable fast actuators do not really destroy the performance of homogeneous high-order sliding-mode controllers. The resulting asymptotic sliding accuracy does not depend on the relative degree of the actuator and is only determined by the sliding order and the actuator time constant. The only exclusion is a rare case, when an asymptotically stable sliding mode $\sigma \equiv 0$ arises with the sliding order being equal to the sum of the system and actuator relative degrees. In such a case the residual chattering gradually disappears, and, though the Theorems are surely still valid, the coefficients a_i can be taken arbitrarily small in Theorem 1. Probably, it is only possible when both relative degrees equal one [13].

One can consider application of smoothing filters at the input of an actuator device, which does not accept discontinuous inputs. If the time constant of the additional artificial actuator is sufficiently small, the resulting actuators will still provide for good performance due to the high sliding order (Fig. 1b).

The most widely used application of high-order sliding modes is based on the artificial increase of the relative degree, when the control derivative is treated as a new control. This results in a smooth control entering an actuator. Also noisy fast stable sensors are to be considered at the output of the system.. Due to the restrictions of the brief paper framework the proof of the HOSM robustness in all these cases will be published in a separate paper.

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