NIVEL BASICO

3.1 For each of the functions $f(x)$ given next, find whether f is (a) continuously lifferentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

(**1**) $f(x) = x^2 + |x|$.
(**2**) $f(x) = x + \text{sgn}(x)$.
(**3**) $f(x) = \sin(x)\ \text{sgn}(x)$.
(**4**) $f(x) = -x + a\sin(x)$.
(**5**) $f(x) = -x + 2|x|$.
(**6**) $f(x) = \tan(x)$.

(**7**) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$.

(**8**) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$.

Nivel Avanzado(notaciones del capitulo de dependencia continua de parámetros)

3.26 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0, t_1] \times D$, for some domain $D \subset R^n$. Let $y(t)$ be a solution of (3.1) on a maximal open interval $[t_0, T) \subset [t_0, t_1]$ with $T < \infty$. Let W be any compact subset of D. Show that there is some $t \in [t_0, T)$ with $y(t) \notin W$.
Hint: Use the previous exercise.

3.27 (**[43]**) Let $x_1 : R \to R^n$ and $x_2 : R \to R^n$ be differentiable functions such that

$$\|x_1(a) - x_2(a)\| \le \gamma, \quad \|\dot{x}_i(t) - f((t, x_i(t))\| \le \mu_i, \text{ for } i = 1,2$$

for $a \le t \le b$. If f satisfies the Lipschitz condition (3.2), show that

$$\|x_1(t) - x_2(t)\| \le \gamma e^{L(t-a)} + (\mu_1 + \mu_2)\left[\frac{e^{L(t-a)} - 1}{L}\right], \quad \text{for } a \le t \le b$$

3.29 Let $f(t, x)$ and its partial derivatives with respect to x be continuous in (t, x) for all $(t, x) \in [t_0, t_1] \times R^n$. Let $x(t, \eta)$ be the solution of (3.1) that starts at $x(t_0) = \eta$ and suppose $x(t, \eta)$ is defined on $[t_0, t_1]$. Show that $x(t, \eta)$ is continuously differentiable with respect to η and find the variational equation satisfied by $[\partial x/\partial \eta]$. Hint: Put $y = x - \eta$ to transform (3.1) into

$$\dot{y} = f(t, y + \eta), \quad y(t_0) = 0$$

with η as a parameter.

3.30 Let $f(t, x)$ and its partial derivative with respect to x be continuous in (t, x) for all $(t, x) \in R \times R^n$. Let $x(t, a, \eta)$ be the solution of (3.1) that starts at $x(a) = \eta$ and suppose that $x(t, a, \eta)$ is defined on $[a, t_1]$. Show that $x(t, a, \eta)$ is continuously differentiable with respect to a and η and let $x_a(t)$ and $x_\eta(t)$ denote $[\partial x/\partial a]$ and $[\partial x/\partial \eta]$, respectively. Show that $x_a(t)$ and $x_\eta(t)$ satisfy the identity

$$x_a(t) + x_\eta(t) f(a, \eta) \equiv 0, \quad \forall\, t \in [a, t_1]$$

3.31 ([43]) Let $f : R \times R \to R$ be a continuous function. Suppose that $f(t, x)$ is locally Lipschitz and nondecreasing in x for each fixed value of t. Let $x(t)$ be a solution of $\dot{x} = f(t, x)$ on an interval $[a, b]$. If a continuous function $y(t)$ satisfies the integral inequality

$$y(t) \le x(a) + \int_a^t f(s, y(s))\, ds$$

for $a \le t \le b$, show that $y(t) \le x(t)$ throughout this interval.