Chapter 14

Sets, Functions and Metric Spaces

14.1 Functions and sets

14.1.1 The function concept

Definition 14.1 *Let us consider two sets $\mathcal{A}$ and $\mathcal{B}$ whose elements may be any objects whatsoever. Suppose that with each element $x \in \mathcal{A}$ there is associated, in some manner, an element $y \in \mathcal{B}$ which we denote by $y = f(x)$.*

1. *Then f is said to be a* ***function*** *from $\mathcal{A}$ to $\mathcal{B}$ or a* ***mapping*** *of $\mathcal{A}$ into $\mathcal{B}$.*

2. *If $\mathcal{E} \subset \mathcal{A}$ then $f(\mathcal{E})$ is defined to be the set of all elements $f(x)$, $x \in \mathcal{E}$ and it is called the image of $\mathcal{E}$ under f. The notations $f(\mathcal{A})$ is called the* ***range*** *of f (evidently, that $f(\mathcal{A}) \subseteq \mathcal{B}$). If $f(\mathcal{A}) = \mathcal{B}$ we say that f maps $\mathcal{A}$ onto $\mathcal{B}$.*

3. *For $\mathcal{D} \subset \mathcal{B}$ the notation $f^{-1}(\mathcal{D})$ denotes the set of all $x \in \mathcal{A}$ such that $f(x) \in \mathcal{B}$. We call $f^{-1}(\mathcal{D})$ the* ***inverse image*** *of $\mathcal{D}$ under f. So, if $y \in \mathcal{D}$ then $f^{-1}(y)$ is the set of all $x \in \mathcal{A}$ such that $f(x) = y$. If for each $y \in \mathcal{B}$ the set $f^{-1}(y)$ consists of at most one element of $\mathcal{A}$ then f is said to be* ***one-to-one mapping*** *of $\mathcal{A}$ to $\mathcal{B}$.*

The one-to-one mapping f means that $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ for any $x_1, x_2 \in \mathcal{A}$. We often will use the following notation for the mapping f:

$$\boxed{f : \mathcal{A} \to \mathcal{B}} \tag{14.1}$$

If, in particular, $\mathcal{A} = \mathbb{R}^n$ and $\mathcal{B} = \mathbb{R}^m$ we will write

$$\boxed{f : \mathbb{R}^n \to \mathbb{R}^m} \tag{14.2}$$

Definition 14.2 *If for two sets $\mathcal{A}$ and $\mathcal{B}$ there exists an one-to-one mapping then we say that these sets are **equivalent** and we write*

$$\boxed{\mathcal{A} \sim \mathcal{B}} \tag{14.3}$$

Claim 14.1 *The relation of equivalency $(\sim)$ clearly has the following properties:*

a) *it is **reflexive**, i.e., $\mathcal{A} \sim \mathcal{A}$;*

b) *it is **symmetric**, i.e., if $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{B} \sim \mathcal{A}$;*

c) *it is **transitive**, i.e., if $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ then $\mathcal{A} \sim \mathcal{C}$.*

14.1.2 Finite, countable and uncountable sets

Denote by $\mathcal{J}_n$ the set of positive numbers $1, 2, ..., n$, that is,

$$\boxed{\mathcal{J}_n = \{1, 2, ..., n\}}$$

and by $\mathcal{J}$ we will denote the set of all positive numbers, namely,

$$\boxed{\mathcal{J} = \{1, 2, ...\}}$$

Definition 14.3 *For any $\mathcal{A}$ we say:*

1. *$\mathcal{A}$ is **finite** if*

$$\mathcal{A} \sim \mathcal{J}_n$$

*for some finite n (the **empty set** $\varnothing$, which does not contain any element, is also considered as finite);*

*2. $\mathcal{A}$ is **countable** (enumerable or denumerable) if*

$$\mathcal{A} \sim \mathcal{J}$$

*3. $\mathcal{A}$ is **uncountable** if it is neither finite nor countable;*

*4. $\mathcal{A}$ is **at most countable** if it is both finite or countable.*

Evidently that if $\mathcal{A}$ is infinite then it is equivalent to one of its subsets. Also it is clear that any infinite subset of a countable set is countable.

Definition 14.4 *By **a sequence** we mean a function f defined on the set $\mathcal{J}$ of all positive integers. If $x_n = f(n)$ it is customarily to denote the corresponding sequence by*

$$\boxed{\{x_n\} := \{x_1, x_2, ...\}}$$

(sometimes, this sequence starts with x_0 but not with x_1).

Claim 14.2

1. *The set $\mathcal{N}$ of all integers is countable;*
2. *The set $\mathcal{Q}$ of all rational numbers is countable;*
3. *The set $\mathbb{R}$ of all real numbers is uncountable.*

14.1.3 Algebra of sets

Definition 14.5 *Let $\mathcal{A}$ and Ω be sets. Suppose that with each element $\alpha \in \mathcal{A}$ there is associated a subset $\mathcal{E}_\alpha \subset \Omega$. Then*

*a) The **union** of the sets $\mathcal{E}_\alpha$ is defined to be the set $\mathcal{S}$ such that $x \in S$ if and only if $x \in \mathcal{E}_\alpha$ at least for one $\alpha \in \mathcal{A}$. It will be denoted by*

$$\boxed{\mathcal{S} := \bigcup_{\alpha \in A} \mathcal{E}_\alpha} \tag{14.4}$$

If $\mathcal{A}$ consists of all integers $(1, 2, ..., n)$, that means, $\mathcal{A} = \mathcal{J}_n$, we will use the notation

$$\boxed{\mathcal{S} := \bigcup_{\alpha=1}^{n} \mathcal{E}_\alpha} \tag{14.5}$$

and if $\mathcal{A}$ consists of all integers $(1, 2, ...)$, that means, $\mathcal{A} = \mathcal{J}$, we will use the notation

$$\boxed{\mathcal{S} := \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha} \tag{14.6}$$

b) *The **intersection** of the sets $\mathcal{E}_\alpha$ is defined as the set $\mathcal{P}$ such that $x \in \mathcal{P}$ if and only if $x \in \mathcal{E}_\alpha$ for every $\alpha \in \mathcal{A}$. It will be denoted by*

$$\boxed{\mathcal{S} := \bigcap_{\alpha \in \mathcal{A}} \mathcal{E}_\alpha} \tag{14.7}$$

If $\mathcal{A}$ consists of all integers $(1, 2, ..., n)$, that means, $\mathcal{A} = \mathcal{J}_n$, we will use the notation

$$\boxed{\mathcal{S} := \bigcap_{\alpha=1}^{n} \mathcal{E}_\alpha} \tag{14.8}$$

and if $\mathcal{A}$ consists of all integers $(1, 2, ...)$, that means, $\mathcal{A} = \mathcal{J}$, we will use the notation

$$\boxed{\mathcal{S} := \bigcap_{\alpha=1}^{\infty} \mathcal{E}_\alpha} \tag{14.9}$$

*If for two sets $\mathcal{A}$ and $\mathcal{B}$ we have $\mathcal{A} \cap \mathcal{B} = \varnothing$, we say that these two sets are **disjoint**.*

c) *The **complement** of $\mathcal{A}$ relative to $\mathcal{B}$, denoted by $\mathcal{B} - \mathcal{A}$, is defined to be the set*

$$\boxed{\mathcal{B} - \mathcal{A} := \{x : x \in \mathcal{B}, \text{ but } x \notin \mathcal{A}\}} \tag{14.10}$$

The sets $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{B} - \mathcal{A}$ are illustrated at Fig.14.1.

Using these graphic illustrations it is possible easily to prove the following set-theoretical identities for union and intersection.

Proposition 14.1

1.

$$\boxed{\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}, \ \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}}$$

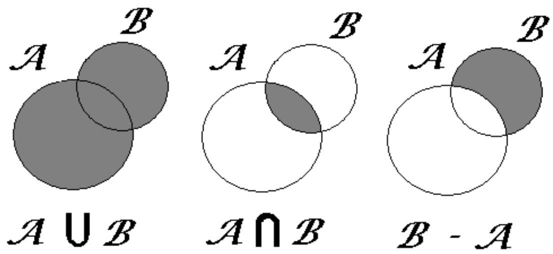

Figure 14.1: Two sets relations.

2.
$$\boxed{\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})}$$

3.
$$\boxed{(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C}) = \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C})}$$

4.
$$\boxed{(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{C} \cap \mathcal{A}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C})}$$

5.
$$\boxed{\mathcal{A} \cap (\mathcal{B} - \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) - (\mathcal{A} \cap \mathcal{C})}$$

6.
$$\boxed{(\mathcal{A} - \mathcal{C}) \cap (\mathcal{B} - \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) - \mathcal{C}}$$

7.
$$(\mathcal{A} - \mathcal{B}) \cup \mathcal{B} = \mathcal{A}$$

if and only if $\mathcal{B} \subseteq \mathcal{A}$.

8.
$$\boxed{\mathcal{A} \subset \mathcal{A} \cup \mathcal{B},\ \mathcal{A} \cap \mathcal{B} \subset \mathcal{A}}$$

9.

$$\boxed{\mathcal{A} \cup \varnothing = \mathcal{A},\ \mathcal{A} \cap \varnothing = \varnothing}$$

10.

$$\boxed{\mathcal{A} \cup \mathcal{B} = \mathcal{B},\ \mathcal{A} \cap \mathcal{B} = \mathcal{A}}$$

if $\mathcal{A} \subset \mathcal{B}$.

The next relations generalize the previous unions and intersections to arbitrary ones.

Proposition 14.2

1. *Let* $f : \mathcal{S} \to \mathcal{T}$ *be a function and* $\mathcal{A}, \mathcal{B}$ *any any subsets of* S. *Then*

$$\boxed{f(\mathcal{A} \cup \mathcal{B}) = f(\mathcal{A}) \cup f(\mathcal{B})}$$

2. *For any* $\mathcal{Y} \subseteq \mathcal{T}$ *define* $f^{-1}(\mathcal{Y})$ *as the largest subset of* S *which* f *maps into* Y. *Then*

a)

$$\boxed{\mathcal{X} \subseteq f^{-1}(f(\mathcal{X}))}$$

b)

$$\boxed{f(f^{-1}(\mathcal{Y})) \subseteq \mathcal{Y}}$$

and

$$\boxed{f(f^{-1}(\mathcal{Y})) = \mathcal{Y}}$$

if and only if $\mathcal{T} = f(\mathcal{S})$.

c)

$$\boxed{f^{-1}(\mathcal{Y}_1 \cup \mathcal{Y}_2) = f^{-1}(\mathcal{Y}_1) \cup f^{-1}(\mathcal{Y}_2)}$$

d)

$$\boxed{f^{-1}(\mathcal{Y}_1 \cap \mathcal{Y}_2) = f^{-1}(\mathcal{Y}_1) \cap f^{-1}(\mathcal{Y}_2)}$$

e)

$$\boxed{f^{-1}(\mathcal{T} - \mathcal{Y}) = \mathcal{S} - f^{-1}(\mathcal{Y})}$$

and for subsets $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{S}$ *it follows that*

$$\boxed{f(\mathcal{A} - \mathcal{B}) = f(\mathcal{A}) - f(\mathcal{B})}$$

14.2 Metric spaces

14.2.1 Metric definition and examples of metrics

Definition 14.6 *A set $\mathcal{X}$, whose elements we shall call points, is said to be a* ***metric space*** *if with any two points p and q of $\mathcal{X}$ there is associated a real number $d(p,q)$, called a* ***distance*** *between p and q, such that*

a)

$$\boxed{\begin{array}{c} d(p,q) > 0 \text{ if } p \neq q \\ d(p,p) = 0 \end{array}} \tag{14.11}$$

b)

$$\boxed{d(p,q) = d(q,p)} \tag{14.12}$$

c) *for any $r \in \mathcal{X}$ the following "triangle inequality" holds:*

$$\boxed{d(p,q) \leq d(p,r) + d(r,q)} \tag{14.13}$$

Any function with these properties is called a ***distance function*** *or a* ***metric.***

Example 14.1 *The following functions are metrics:*

1. *For any p, q from the* ***Euclidian space*** $\mathbb{R}^n$

   a) *the* ***Euclidian metric:***

$$\boxed{d(p,q) = \|p - q\|} \tag{14.14}$$

   b) *the* ***discrete metric:***

$$\boxed{d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q \end{cases}} \tag{14.15}$$

   c) *the* ***weighted metric:***

$$\boxed{\begin{array}{c} d(p,q) = \|p - q\|_Q := \sqrt{(p-q)^{\mathsf{T}} Q (p-q)} \\ Q = Q^{\mathsf{T}} > 0 \end{array}} \tag{14.16}$$

d) *the **module metric**:*

$$\boxed{d(p,q) = \sum_{i=1}^{n} |p_i - q_i|} \tag{14.17}$$

e) *the **Chebyshev's metric**:*

$$\boxed{d(p,q) = \max\{|p_1 - q_1|, ..., |p_n - q_n|\}} \tag{14.18}$$

f) *the **Prokhorov's metric**:*

$$\boxed{d(p,q) = \frac{\|p-q\|}{1+\|p-q\|} \in [0,1)} \tag{14.19}$$

*2. For any $z_1$and z_2 of the **complex plane** $\mathbb{C}$*

$$\boxed{\begin{array}{c} d(z_1,z_2) = |z_1 - z_2| = \\ \sqrt{(\mathrm{Re}(z_1-z_2))^2 + (\mathrm{Im}(z_1-z_2))^2} \end{array}} \tag{14.20}$$

14.2.2 Set structures

Let $\mathcal{X}$ be a metric space. All points and sets mentioned below will be understood to be elements and subsets of $\mathcal{X}$.

Definition 14.7

*a) A **neighborhood** of a point x is a set $\mathcal{N}_r(x)$ consisting of all points y such that $d(x,y) < r$ where the number r is called the radius of $\mathcal{N}_r(x)$, that is,*

$$\boxed{\mathcal{N}_r(x) := \{x \in \mathcal{X} : d(x,y) < r\}} \tag{14.21}$$

*b) A point $x \in \mathcal{X}$ is a **limit point** of the set $\mathcal{E} \subset \mathcal{X}$ if every neighborhood of x contains a point $y \neq x$ such that $q \in \mathcal{E}$.*

*c) If $x \in \mathcal{E}$ and x is not a limit point of then x is called an **isolated point** of $\mathcal{E}$.*

*d) $\mathcal{E} \subset \mathcal{X}$ is **closed** if every limit of $\mathcal{E}$ is a point of $\mathcal{E}$.*

e) *A point* $x \in \mathcal{E}$ *is an* ***interior point*** *of* $\mathcal{E}$ *if there is a neighborhood of* $\mathcal{N}_r(x)$ *of* x *such that* $\mathcal{N}_r(x) \subset \mathcal{E}$.

f) $\mathcal{E}$ *is* ***open*** *if every point of* $\mathcal{E}$ *is an interior point of* $\mathcal{E}$.

g) *The* ***complement*** $\mathcal{E}^c$ *of* $\mathcal{E}$ *is the set of all points* $x \in \mathcal{X}$ *such that* $x \notin \mathcal{E}$.

h) $\mathcal{E}$ *is* ***bounded*** *if there exists a real number* M *and a point* $x \in \mathcal{E}$ *such that* $d(x, y) < M$ *for all* $y \in \mathcal{E}$.

i) $\mathcal{E}$ *is* ***dense*** *in* $\mathcal{X}$ *if every point* $x \in \mathcal{X}$ *is a limit point of* $\mathcal{E}$, *or a point of* $\mathcal{E}$, *or both.*

j) $\mathcal{E}$ *is* ***connected*** *in* $\mathcal{X}$ *if it is not a union of two nonempty* ***separated sets****, that is,* $\mathcal{E}$ *can not be represented as* $\mathcal{E} = \mathcal{A} \cup \mathcal{B}$ *where* $\mathcal{A} \neq \varnothing$, $\mathcal{B} \neq \varnothing$ *and* $\mathcal{A} \cap \mathcal{B} = \varnothing$.

Example 14.2 *The set* $J_{open}(p)$ *defined as*

$$\boxed{J_{open} := \{x \in \mathcal{X},\ d(x, p) < r\}}$$

is an open set but the set $J_{closed}(p)$ *defined as*

$$\boxed{J_{closed}(p) := \{x \in \mathcal{X},\ d(x, p) \leq r\}}$$

is closed.

The following claims seem to be evident and, that's why, they are given without proofs.

Claim 14.3

1. *Every neighborhood* $\mathcal{N}_r(x) \subset \mathcal{E}$ *is an open set.*
2. *If* x *is a limit point of* $\mathcal{E}$ *then every neighborhood* $\mathcal{N}_r(x) \subset \mathcal{E}$ *contains infinitely many points of* $\mathcal{E}$.
3. *A finite point set has no limit points.*

Let us prove the following lemma concerning complement sets.

Lemma 14.1 *Let $\{\mathcal{E}_\alpha\}$ be a collection (finite or infinite) of sets $\mathcal{E}_\alpha \subseteq \mathcal{X}$. Then*

$$\boxed{\left(\bigcup_\alpha \mathcal{E}_\alpha\right)^c = \bigcap_\alpha \mathcal{E}_\alpha^c} \tag{14.22}$$

Proof. If $x \in \left(\bigcup_\alpha \mathcal{E}_\alpha\right)^c$ then, evidently, $x \notin \bigcup_\alpha \mathcal{E}_\alpha$ and, hence, $x \notin \mathcal{E}_\alpha$ for any α. This means that $x \in \bigcap_\alpha \mathcal{E}_\alpha^c$. Thus,

$$\left(\bigcup_\alpha \mathcal{E}_\alpha\right)^c \subseteq \bigcap_\alpha \mathcal{E}_\alpha^c \tag{14.23}$$

Conversely, if $x \in \bigcap_\alpha \mathcal{E}_\alpha^c$ then $x \in \mathcal{E}_\alpha^c$ for every α and, hence, $x \notin \bigcup_\alpha \mathcal{E}_\alpha$. So, $x \in \left(\bigcup_\alpha \mathcal{E}_\alpha\right)^c$ that implies

$$\bigcap_\alpha \mathcal{E}_\alpha^c \subseteq \left(\bigcup_\alpha \mathcal{E}_\alpha\right)^c \tag{14.24}$$

Combining (14.23) and (14.24) gives (14.22). Lemma is proven. ■

This lemma provides the following corollaries.

Corollary 14.1

a) A set $\mathcal{E}$ is open if and only if its complement $\mathcal{E}^c$ is closed.

b) A set $\mathcal{E}$ is closed if and only if its complement $\mathcal{E}^c$ is open.

c) For any collection $\{\mathcal{E}_\alpha\}$ of open sets $\mathcal{E}_\alpha$ the set $\bigcup_\alpha \mathcal{E}_\alpha$ is open.

d) For any collection $\{\mathcal{E}_\alpha\}$ of closed sets $\mathcal{E}_\alpha$ the set $\bigcap_\alpha \mathcal{E}_\alpha^c$ is closed.

e) For any finite collection $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ of open sets $\mathcal{E}_\alpha$ the set $\bigcap_\alpha \mathcal{E}_\alpha^c$ is open too.

f) For any finite collection $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ of closed sets $\mathcal{E}_\alpha$ the set $\bigcup_\alpha \mathcal{E}_\alpha$ is closed too.

Definition 14.8 *Let $\mathcal{X}$ be a metric space and $\mathcal{E} \subset \mathcal{X}$. Denote by $\mathcal{E}'$ the set of all limit points of $\mathcal{E}$. Then the set* cl$\mathcal{E}$ *defined as*

$$\boxed{\mathrm{cl}\mathcal{E} := \mathcal{E} \cup \mathcal{E}'} \tag{14.25}$$

is called the ***closure*** *or $\mathcal{E}$.*

The next properties seem to be logical consequences of this definition.

Proposition 14.3 *If $\mathcal{X}$ be a metric space and $\mathcal{E} \subset \mathcal{X}$, then*

a) cl$\mathcal{E}$ *is closed;*

b) $\mathcal{E} = \mathrm{cl}\mathcal{E}$ *if and only if $\mathcal{E}$ is closed;*

c) $\mathrm{cl}\mathcal{E} \subset \mathcal{P}$ *for every closed set $\mathcal{P} \subset \mathcal{X}$ such that $\mathcal{E} \subset \mathcal{P}$;*

d) If is a nonempty set of real numbers which is bounded above, i.e., $\varnothing \neq \mathcal{E} \subset \mathbb{R}$ and $y := \sup \mathcal{E} < \infty$. Then $y \in \mathrm{cl}\mathcal{E}$ and, hence, $y \in \mathcal{E}$ if $\mathcal{E}$ is closed.

Proof.
a) If $x \in \mathcal{X}$ and $y \notin \mathrm{cl}\mathcal{E}$ then x is neither a point of $\mathcal{E}$ nor a limit point of $\mathcal{E}$. Hence x has a neighborhood which does not intersect $\mathcal{E}$. Therefore the complement $\mathcal{E}^c$ of $\mathcal{E}$ is an open set. So, cl$\mathcal{E}$ is closed.
b) If $\mathcal{E} = \mathrm{cl}\mathcal{E}$ then by *a)* it follows that $\mathcal{E}$ is closed. If $\mathcal{E}$ is closed then for $\mathcal{E}'$, defined in (14.8), we have that $\mathcal{E}' \subset \mathcal{E}$. Hence, $\mathcal{E} = \mathrm{cl}\mathcal{E}$.
c) $\mathcal{P}$ is closed and $\mathcal{P} \supset \mathcal{E}$ (defined in (14.8)) then $\mathcal{P} \supset \mathcal{P}'$ and, hence, $\mathcal{P} \supset \mathcal{E}'$. Thus $\mathcal{P} \supset \mathrm{cl}\mathcal{E}$.
d) If $y \in \mathcal{E}$ then $y \in \mathrm{cl}\mathcal{E}$. Assume $y \notin \mathcal{E}$. Then for any $\varepsilon > 0$ there exists a point $x \in \mathcal{E}$ such that $y - \varepsilon < x < y$, for otherwise $(y - \varepsilon)$ would be an upper bound of $\mathcal{E}$ that contradicts to the supposition $\sup \mathcal{E} = y$. Thus y is a limit point of $\mathcal{E}$. Hence, $y \in \mathrm{cl}\mathcal{E}$.
The proposition is proven. ■

Definition 14.9 *Let $\mathcal{E}$ be a set of a metric space $\mathcal{X}$. A point $x \in \mathcal{E}$ is called* ***a boundary point*** *of $\mathcal{E}$ if any neighborhood $\mathcal{N}_r(x)$ of this point contains at least one point of $\mathcal{E}$ and at least one point of $\mathcal{X} - \mathcal{E}$. The set of all boundary points of $\mathcal{E}$ is called the* ***boundary of the set*** *$\mathcal{E}$ and is denoted by $\partial\mathcal{E}$.*

It is not difficult to verify that

$$\boxed{\partial\mathcal{E} = \mathrm{cl}\mathcal{E} \cap \mathrm{cl}(\mathcal{X} - \mathcal{E})} \tag{14.26}$$

Denoting by

$$\boxed{\mathrm{int}\mathcal{E} := \mathcal{E} - \partial\mathcal{E}} \tag{14.27}$$

the set of all internal points of the set $\mathcal{E}$, it is easily verify that

$$\boxed{\begin{array}{c} \mathrm{int}\mathcal{E} = \mathcal{X} - \mathrm{cl}\,(\mathcal{X} - \mathcal{E}) \\ \mathrm{int}\,(\mathcal{X} - \mathcal{E}) = \mathcal{X} - \mathrm{cl}\mathcal{E} \\ \mathrm{int}\,(\mathrm{int}\mathcal{E}) = \mathrm{int}\mathcal{E} \\ \text{If } \mathrm{cl}\mathcal{E} \cap \mathrm{cl}\mathcal{D} = \varnothing \text{ then } \partial\,(\mathcal{E} \cup \mathcal{D}) = \partial\mathcal{E} \cup \partial\mathcal{D} \end{array}} \tag{14.28}$$

14.2.3 Compact sets

Definition 14.10

1. *By an* ***open cover of a set*** $\mathcal{E}$ *in a metric space* $\mathcal{X}$ *we mean a collection* $\{\mathcal{G}_\alpha\}$ *of open subsets of* $\mathcal{X}$ *such that*

$$\boxed{\mathcal{E} \subset \bigcup_\alpha \mathcal{G}_\alpha} \tag{14.29}$$

2. *A subset* $\mathcal{K}$ *of a metric space* $\mathcal{X}$ *is said to be* ***compact*** *if every open cover of* $\mathcal{K}$ *contains a finite subcover, more exactly, there are a finite number of indices* $\alpha_1, ..., \alpha_n$ *such that*

$$\boxed{\mathcal{E} \subset \mathcal{G}_{\alpha_1} \cup \cdots \cup \mathcal{G}_{\alpha_n}} \tag{14.30}$$

Remark 14.1 *Evidently that every finite set is compact.*

Theorem 14.1 *A set* $\mathcal{K} \subset \mathcal{Y} \subset \mathcal{X}$ *is a compact relative to* $\mathcal{X}$ *if and only if* $\mathcal{K}$ *is a compact relative to* $\mathcal{Y}$.

Proof. *Necessity.* Suppose $\mathcal{K}$ is a compact relative to $\mathcal{X}$. Hence, by the definition (14.30) there exists its finite subcover such that

$$\mathcal{K} \subset \mathcal{G}_{\alpha_1} \cup \cdots \cup \mathcal{G}_{\alpha_n} \tag{14.31}$$

where $\mathcal{G}_{\alpha_i}$ is an open set with respect to $\mathcal{X}$. On the other hand $\mathcal{K} \subset \bigcup_{\alpha} \mathcal{V}_{\alpha}$ where $\{\mathcal{V}_{\alpha}\}$ is a collection of sets open with respect to $\mathcal{Y}$. But any open set $\mathcal{V}_{\alpha}$ can be represented as $\mathcal{V}_{\alpha} = \mathcal{Y} \cap \mathcal{G}_{\alpha}$. So, (14.31) implies

$$\mathcal{K} \subset \mathcal{V}_{\alpha_1} \cup \cdots \cup \mathcal{V}_{\alpha_n} \tag{14.32}$$

Sufficiency. Conversely, if $\mathcal{K}$ is a compact relative to $\mathcal{Y}$ then there exists a finite collection $\{\mathcal{V}_{\alpha}\}$ of open sets in $\mathcal{Y}$ such that (14.32) holds. Putting $\mathcal{V}_{\alpha} = \mathcal{Y} \cap \mathcal{G}_{\alpha}$ for a special choice of indices $\alpha_1, ..., \alpha_n$ it follows that $\mathcal{V}_{\alpha} \subset \mathcal{G}_{\alpha}$ that implies (14.31). Theorem is proven. ■

Theorem 14.2 *Compact sets of metric spaces are closed.*

Proof. Suppose $\mathcal{K}$ is a compact subset of a metric space $\mathcal{X}$. Let $x \in \mathcal{X}$ but $x \notin \mathcal{K}$ and $y \in \mathcal{K}$. Consider the neighborhoods $\mathcal{N}_r(x)$ $\mathcal{N}_r(y)$ of these points with $r < \frac{1}{2} d(x, y)$. Since $\mathcal{K}$ is a compact there are finitely many points $y_1, ..., y_n$ such that

$$\mathcal{K} \subset \mathcal{N}_r(y_1) \cup \cdots \cup \mathcal{N}_r(y_n) = \mathcal{N}$$

If $\mathcal{V} = \mathcal{N}_{r_1}(x) \cap \cdots \cap \mathcal{N}_{r_n}(x)$, then evidently $\mathcal{V}$ is a neighborhood of x which does not intersect $\mathcal{N}$ and, hence, $\mathcal{V} \subset \mathcal{K}^c$. So, x is an interior point of $\mathcal{K}^c$. Theorem is proven. ■

The following two propositions seem to be evident.

Proposition 14.4

1. *Closed subsets of compact sets are compacts too.*
2. *If $\mathcal{F}$ is closed and $\mathcal{K}$ is compact then $\mathcal{F} \cap \mathcal{K}$ is compact.*

Theorem 14.3 *If $\mathcal{E}$ is an infinite subset of a compact set $\mathcal{K}$ then $\mathcal{E}$ has a limit point in $\mathcal{K}$.*

Proof. If no point of $\mathcal{K}$ were a limit point of $\mathcal{E}$ then $y \in \mathcal{K}$ would have a neighborhood $\mathcal{N}_r(y)$ which contains at most one point of $\mathcal{E}$ (namely, y if $y \in \mathcal{E}$). It is clear that no finite subcollection $\{\mathcal{N}_{r_k}(y)\}$ can cover $\mathcal{E}$. The same is true of $\mathcal{K}$ since $\mathcal{E} \subset \mathcal{K}$. But this contradicts the compactness of $\mathcal{K}$. Theorem is proven. ■

The next theorem explains the compactness property especially in $\mathbb{R}^n$ and is often applied in a control theory analysis.

Theorem 14.4 *If a set $\mathcal{E} \subset \mathbb{R}^n$ then the following three properties are equivalent:*

a) $\mathcal{E}$ is closed and bounded.

b) $\mathcal{E}$ is compact.

c) Every infinite subset of $\mathcal{E}$ has a limit point in $\mathcal{E}$.

Proof. It is the consequence of all previous theorems and propositions and stay for readers consideration. The details of the proof can be found in Chapter 2 of (Rudin 1976). ■

Remark 14.2 *Notice that properties b) and c) are equivalent in any metric space, but a) not.*

14.2.4 Convergent sequences in metric spaces

Convergence

Definition 14.11 *A sequence $\{x_n\}$ in a metric space $\mathcal{X}$ is said to* ***converge*** *if there is a point $x \in \mathcal{X}$ for which for any $\varepsilon > 0$ there exists an integer n_ε such that $n \geq n_\varepsilon$ implies that $d(x_n, x) < \varepsilon$. Here $d(x_n, x)$ is the metric (distance) in $\mathcal{X}$. In this case we say that $\{x_n\}$ converges to x, or that x is a limit of $\{x_n\}$, and we write*

$$\boxed{\lim_{n\to\infty} x_n = x \text{ or } x_n \underset{n\to\infty}{\to} x} \tag{14.33}$$

If $\{x_n\}$ does not converge, it is usually said to ***diverge****.*

Example 14.3 *The sequence $\{1/n\}$ converge to 0 in $\mathbb{R}$, but fails to converge in $\mathbb{R}_+ := \{x \in \mathbb{R} \,| x > 0\}$.*

Theorem 14.5 *Let $\{x_n\}$ be a sequence in a metric space $\mathcal{X}$.*

1. *$\{x_n\}$ converges to $x \in \mathcal{X}$ if and only if every neighborhood $\mathcal{N}_\varepsilon(x)$ of x contains all but (excluding) finitely many of the terms of $\{x_n\}$.*

2. If $x', x'' \in \mathcal{X}$ *and*

$$\boxed{x_n \underset{n\to\infty}{\to} x' \text{ and } x_n \underset{n\to\infty}{\to} x''}$$

then

$$\boxed{x' = x''}$$

3. If $\{x_n\}$ *converges then* $\{x_n\}$ *is bounded.*

4. If $\mathcal{E} \subset \mathcal{X}$ *and* x *is a limit point of* $\mathcal{E}$ *then there is a sequence* $\{x_n\}$ *in* $\mathcal{E}$ *such that* $x = \lim\limits_{n\to\infty} x_n$.

Proof.

1. *a) Necessity.* Suppose $x_n \underset{n\to\infty}{\to} x$ and let $\mathcal{N}_\varepsilon(x)$ (for some $\varepsilon > 0$) be a neighborhood of x. The conditions $d(y, x) < \varepsilon$, $y \in \mathcal{X}$ imply $y \in \mathcal{N}_\varepsilon(x)$. Corresponding to this ε there exists a number n_ε such that for any $n \geq n_\varepsilon$ it follows that $d(x_n, x) < \varepsilon$. Thus, $x_n \in \mathcal{N}_\varepsilon(x)$. So, all x_n are bounded.

   b) Sufficiency. Conversely, suppose every neighborhood of x contains all but finitely many of the terms of $\{x_n\}$. Fixing $\varepsilon > 0$ denoting by $\mathcal{N}_\varepsilon(x)$ the set of all $y \in \mathcal{X}$ such that $d(y, x) < \varepsilon$. By the assumption there exists n_ε such that for any $n \geq n_\varepsilon$ it follows that $x_n \in \mathcal{N}_\varepsilon(x)$. Thus $d(x_n, x) < \varepsilon$ if $n \geq n_\varepsilon$ and, hence, $x_n \underset{n\to\infty}{\to} x$.

2. For the given $\varepsilon > 0$ there exist integers n' and n'' such that $n \geq n'$ implies $d(x_n, x') < \varepsilon/2$ and $n \geq n''$ implies $d(x_n, x'') < \varepsilon/2$. So, for $n \geq \max\{n', n''\}$ it follows $d(x', x'') \leq d(x', x_n) + d(x_n, x'') < \varepsilon$. Taking ε small enough we conclude that $d(x', x'') = 0$.

3. Suppose $x_n \underset{n\to\infty}{\to} x$. Then, evidently there exists an integer n_0 such that for all $n \geq n_0$ we have that $d(x_n, x) < 1$. Define $r := \max\{1, d(x_1, x), ..., d(x_{n_0}, x)\}$. Then $d(x_n, x) < r$ for all $n = 1, 2, ...$

4. For any integer $n = 1, 2, ...$ there exists a point $x_n \in \mathcal{E}$ such that $d(x_n, x) < 1/n$. For any given $\varepsilon > 0$ define n_ε such that $\varepsilon n_\varepsilon > 1$. Then for $n \geq n_\varepsilon$ one has $d(x_n, x) < 1/n < \varepsilon$ that means that $x_n \underset{n\to\infty}{\to} x$.

This completes the proof. ■

Subsequences

Definition 14.12 *Given a sequence* $\{x_n\}$ *let us consider a sequence* $\{n_k\}$ *of positive integers satisfying* $n_1 < n_2 < \cdots$. *Then the sequence* $\{x_{n_k}\}$ *is called a* ***subsequence*** *of* $\{x_n\}$.

Claim 14.4 *If a sequence* $\{x_n\}$ *converges to* x *then any subsequence* $\{x_{n_k}\}$*of* $\{x_n\}$ *converges to the same limit point* x.

Proof. This result can be easily proven by contradiction. Indeed, assuming that two different subsequences $\{x_{n_k}\}$ and $\left\{x_{n_j}\right\}$ have different limit points x' and x'', then it follows that there exists $0 < \varepsilon < d(x', x'')$ and a number k_ε such that for all $k \geq k_\varepsilon$ we shall have: $d\left(x_{n_k}, x_{n_j}\right) > \varepsilon$ that is in the contradiction with the assumption that $\{x_n\}$ converges. ■

Theorem 14.6

a) *If* $\{x_n\}$ *is a sequence in a compact metric space* $\mathcal{X}$ *then it obligatory contains some subsequence* $\{x_{n_k}\}$ *convergent to a point of* $\mathcal{X}$.

b) *Any bounded sequence in* $\mathbb{R}^n$ *contains a convergent subsequent.*

Proof.

a) Let $\mathcal{E}$ be the range of $\{x_n\}$. If $\{x_n\}$ converges then the desired subsequence is this sequence itself. Suppose that $\{x_n\}$ diverges. If $\mathcal{E}$ is finite then obligatory there is a point $x \in \mathcal{E}$ and numbers $n_1 < n_2 < \cdots$ such that $x_1 = x_2 = \cdots = x$. The subsequence $\{x_{n_k}\}$ so obtained converges evidently to x. If $\mathcal{E}$ is infinite then by Theorem (14.3) $\mathcal{E}$ has a limit point $x \in \mathcal{X}$. Choose n_1 so that $d(x_{n_1}, x) < 1$, and, hence, there are integer $n_i > n_{i-1}$ such that $d(x_{n_i}, x) < 1/i$. This means that x_{n_i} converges to x.

b) This follows from a) since Theorem (14.4) implies that every bounded subset of $\mathbb{R}^n$ lies in a compact sunset of $\mathbb{R}^n$.

Theorem is proven. ■

Cauchy sequences

Definition 14.13 *A sequence $\{x_n\}$ in a metric space $\mathcal{X}$ is said to be a* ***Cauchy (fundamental) sequence*** *if for every $\varepsilon > 0$ there is an integer n_ε such that $d(x_n, x_m) < \varepsilon$ if both $n \geq n_\varepsilon$ and $m \geq n_\varepsilon$.*

Defining the *diameter* of $\mathcal{E}$ as

$$\boxed{\operatorname{diam} \mathcal{E} := \sup_{x,y \in \mathcal{E}} d(x, y)} \tag{14.34}$$

one may conclude that if $\mathcal{E}_{n_\varepsilon}$ consists of the points $\{x_{n_\varepsilon}, x_{n_\varepsilon+1}, ...\}$ then $\{x_n\}$ is a Cauchy sequence if and only if

$$\boxed{\lim_{n_\varepsilon \to \infty} \operatorname{diam} \mathcal{E} = 0} \tag{14.35}$$

Theorem 14.7

a) If $\operatorname{cl}\mathcal{E}$ *is the closure of a set $\mathcal{E}$ in a metric space $\mathcal{X}$ then*

$$\boxed{\operatorname{diam} \mathcal{E} = \operatorname{diam} \operatorname{cl}\mathcal{E}} \tag{14.36}$$

b) If $\mathcal{K}_n$ is a sequence of compact sets in $\mathcal{X}$ such that $\mathcal{K}_n \supset \mathcal{K}_{n-1}$ $(n = 2, 3, ...)$ then the set $\mathcal{K} := \bigcap_{n=1}^{\infty} \mathcal{K}_n$ consists exactly of one point.

Proof.

a) Since $\mathcal{E} \subseteq \operatorname{cl}\mathcal{E}$ it follows that

$$\operatorname{diam}\mathcal{E} \leq \operatorname{diam} \operatorname{cl}\mathcal{E} \tag{14.37}$$

Fix $\varepsilon > 0$ and select $x, y \in \operatorname{cl}\mathcal{E}$. By the definition (14.25) there are to points $x', y' \in \mathcal{E}$ such that both $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$ that implies

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) <$$
$$2\varepsilon + d(x', y') \leq 2\varepsilon + \operatorname{diam}\mathcal{E}$$

As the result, we have

$$\operatorname{diam} \operatorname{cl}\mathcal{E} \leq 2\varepsilon + \operatorname{diam}\mathcal{E}$$

and since ε is arbitrary it follows that

$$\text{diam cl}\mathcal{E} \leq \text{diam}\mathcal{E} \tag{14.38}$$

The inequalities (14.37) and (14.38) give (14.36).

b) If $\mathcal{K}$ contains more then one point then diam $\mathcal{K} > 0$. But for each n we have that $\mathcal{K}_n \supset \mathcal{K}$, so that diam $\mathcal{K}_n \geq$ diam $\mathcal{K}$. This contradict that diam $\mathcal{K}_n \underset{n\to\infty}{\to} 0$.

Theorem is proven. ■

The next theorem explains the importance of fundamental sequence in the analysis of metric spaces.

Theorem 14.8

a) Every convergent sequence $\{x_n\}$ given in a metric space $\mathcal{X}$ is a Cauchy sequence.

b) If $\mathcal{X}$ is a compact metric space and if $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$ then $\{x_n\}$ converges to some point in $\mathcal{X}$.

c) In $\mathbb{R}^n$ a sequence converges if and only if it is a Cauchy sequence.

Usually, the claim *c)* is referred to as the **Cauchy criterion**.

Proof.

a) If $x_n \to x$ then for any $\varepsilon > 0$ there exists an integer n_ε such that $d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$. So, $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < 2\varepsilon$ if $n, m \geq n_\varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence.

b) Let $\{x_n\}$ be a Cauchy sequence and the set $\mathcal{E}_{n_\varepsilon}$ contains the points $x_{n_\varepsilon}, x_{n_\varepsilon+1}, x_{n_\varepsilon+2}, \ldots$. Then by Theorem (14.7) and in view of (14.35) and (14.36)

$$\lim_{n_\varepsilon\to\infty} \text{diam cl}\mathcal{E}_{n_\varepsilon} = \lim_{n_\varepsilon\to\infty} \text{diam } \mathcal{E}_{n_\varepsilon} = 0 \tag{14.39}$$

Being a closed subset of the compact space $\mathcal{X}$ each cl$\mathcal{E}_{n_\varepsilon}$ is compact (see Proposition 14.4). And since $\mathcal{E}_n \supset \mathcal{E}_{n+1}$ then cl$\mathcal{E}_n \supset$ cl$\mathcal{E}_{n+1}$. By Theorem 14.7 b), there is a unique point $x \in \mathcal{X}$ which lies in cl$\mathcal{E}_n$. The expression (14.39) means that for any $\varepsilon > 0$ there exists an integer n_ε

such that diam cl$\mathcal{E}_n < \varepsilon$ if $n \geq n_\varepsilon$. Since $x \in$ cl$\mathcal{E}_n$ then $d(x, y) < \varepsilon$ for any $x \in$ cl$\mathcal{E}_n$ that equivalent to the following: $d(x, x_n) < \varepsilon$ if $n \geq n_\varepsilon$. But this means that $x_n \to x$.

c) Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^n$ and define $\mathcal{E}_{n_\varepsilon}$ like in the statement b) but with $\mathbf{x}_n \in \mathbb{R}^n$ instead of x_n. For some n_ε we have that diam $\mathcal{E}_{n_\varepsilon} < 1$. The range of $\{\mathbf{x}_n\}$ is the union of $\mathcal{E}_n$ and the finite set $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{n_\varepsilon - 1}\}$. Hence, $\{\mathbf{x}_n\}$ is bounded and since every bounded sunset in $\mathbb{R}^n$ has a compact closure in $\mathbb{R}^n$, the statement follows from the statement b).

Theorem is proven. ■

Definition 14.14 *A metric space where each Cauchy sequence converges is said to be* ***complete****.*

Example 14.4

1. *By Theorem (14.8) it follows that all Euclidean spaces are complete.*

2. *The space of all rational numbers with the metric $d(x, y) = |x - y|$ is not complete.*

3. *In $\mathbb{R}^n$ any convergent sequence is bounded but not any bounded sequence obligatory converges.*

There is a special case when bounded sequence obligatory converges. Next theorem specifies such sequences.

Theorem 14.9 (Weierstrass theorem) *Any* ***monotonic sequence*** *$\{s_n\}$ of real numbers, namely, when*

a) *$\{s_n\}$ is* ***monotonically non-decreasing****: $s_n \leq s_{n+1}$;*

b) *$\{s_n\}$ is* ***monotonically non-increasing****: $s_n \geq s_{n+1}$;*

converges if and only if it is bounded.

Proof. If $\{s_n\}$ converges it is bounded by Theorem (14.5) the claim 3. Suppose that and $\{s_n\}$ is bounded, namely, $\sup s_n = s < \infty$. Then $s_n \leq s$ and for every $\varepsilon > 0$ there exists an integer n_ε such that $s - \varepsilon \leq s_n \leq s$ for otherwise $s - \varepsilon$ would be an upper bound for $\{s_n\}$. Since $\{s_n\}$ increases and ε is arbitrary small this means $s_n \to s$. The case $s_n \geq s_{n+1}$ is considered analogously. Theorem is proven. ■

Upper and lower limits in $\mathbb{R}$

Definition 14.15 *Let $\{s_n\}$ be a sequence of real numbers in $\mathbb{R}$.*

a) *If for every real M there exists an integer n_M such that $s_n \geq M$ for all $n \geq n_M$ we then write*

$$\boxed{s_n \to \infty} \tag{14.40}$$

b) *If for every real M there exists an integer n_M such that $s_n \leq M$ for all $n \geq n_M$ we then write*

$$\boxed{s_n \to -\infty} \tag{14.41}$$

c) *Define the **upper limit** of a sequence $\{s_n\}$ as*

$$\boxed{\limsup_{n\to\infty} s_n := \lim_{t\to\infty} \sup_{n\geq t} s_n} \tag{14.42}$$

which may be treated as a biggest limit of all possible subsequences.

d) *Define the **lower limit** of a sequence $\{s_n\}$ as*

$$\boxed{\liminf_{n\to\infty} s_n := \lim_{t\to\infty} \inf_{n\geq t} s_n} \tag{14.43}$$

which may be treated as a lowest limit of all possible subsequences.

The following theorem whose proof is quit trivial is often used in many practical problems.

Theorem 14.10 *Let $\{s_n\}$ and $\{t_n\}$ be two sequences of real numbers in $\mathbb{R}$. Then the following properties hold:*

1.

$$\boxed{\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n} \tag{14.44}$$

2.

$$\boxed{\begin{array}{lll} \limsup\limits_{n\to\infty} s_n = \infty & \textit{if} & s_n \to \infty \\ \liminf\limits_{n\to\infty} s_n = -\infty & \textit{if} & s_n \to -\infty \end{array}} \tag{14.45}$$

3.

$$\boxed{\limsup_{n\to\infty}(s_n + t_n) \le \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n} \tag{14.46}$$

4.

$$\boxed{\liminf_{n\to\infty}(s_n + t_n) \ge \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n} \tag{14.47}$$

5. *If* $\lim\limits_{n\to\infty} s_n = s$ *then*

$$\boxed{\liminf_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s} \tag{14.48}$$

6. *If* $s_n \le t_n$ *for all* $n \ge M$ *which is fixed then*

$$\boxed{\begin{array}{c} \limsup\limits_{n\to\infty} s_n \le \limsup\limits_{n\to\infty} t_n \\ \liminf\limits_{n\to\infty} s_n \le \liminf\limits_{n\to\infty} t_n \end{array}} \tag{14.49}$$

Example 14.5

1.

$$\limsup_{n\to\infty} \sin\left(\frac{\pi}{2}n\right) = 1, \ \liminf_{n\to\infty} \sin\left(\frac{\pi}{2}n\right) = -1$$

2.

$$\limsup_{n\to\infty} \tan\left(\frac{\pi}{2}n\right) = \infty, \ \liminf_{n\to\infty} \tan\left(\frac{\pi}{2}n\right) = -\infty$$

3. *For* $s_n = \dfrac{(-1)^n}{1 + 1/n}$

$$\limsup_{n\to\infty} s_n = 1, \ \liminf_{n\to\infty} s_n = -1$$

14.2.5 Continuity and function limits in metric spaces

Continuity and limits of functions

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $\mathcal{E} \subset \mathcal{X}$, f maps $\mathcal{E}$ into $\mathcal{Y}$ and $p \in \mathcal{X}$.

Definition 14.16

a) *We write*

$$\boxed{\lim_{x \to p} f(x) = q} \tag{14.50}$$

if there is a point $q \in \mathcal{Y}$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon, p) > 0$ *for which* $d_{\mathcal{Y}}(f(x), q) < \varepsilon$ *for all* $x \in \mathcal{E}$ *for which* $d_{\mathcal{X}}(x, p) < \delta$. *The symbols* $d_{\mathcal{Y}}$ *and* $d_{\mathcal{X}}$ *are referred to as the distance in* $\mathcal{X}$ *and* $\mathcal{Y}$, *respectively. Notice that* f *may be not defined at* p *since* p *may not belong to* $\mathcal{E}$.

b) *If, in addition,* $p \in \mathcal{E}$ *and* $d_{\mathcal{Y}}(f(x), f(p)) < \varepsilon$ *for every* $\varepsilon > 0$ *and for all* $x \in \mathcal{E}$ *for which* $d_{\mathcal{X}}(x, p) < \delta = \delta(\varepsilon)$ *then* f *is said to be* ***continuous*** *at the point* p.

c) *If* f *is continuous at every point of* $\mathcal{E}$ *then* f *is said to be* ***continuous on*** $\mathcal{E}$.

d) *If for any* $x, y \in \mathcal{E} \subseteq \mathcal{X}$

$$\boxed{d_{\mathcal{Y}}(f(x), f(y)) \le L_f d_{\mathcal{X}}(x, y), \ L_f < \infty} \tag{14.51}$$

then f *is said to be* ***Lipschitz continuous on*** $\mathcal{E}$.

Remark 14.3 *If* p *is a limit point of* $\mathcal{E}$ *then* f *is continuous at the point* p *if and only if*

$$\boxed{\lim_{x \to p} f(x) = f(p)} \tag{14.52}$$

The proof of this result follows directly from the definition above.

The following properties related to continuity are evidently fulfilled.

Proposition 14.5

1. *If for metric spaces* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *the following mappings are defined:*

$$f : \mathcal{E} \subset \mathcal{X} \to \mathcal{Y}, \ g : f(\mathcal{E}) \to \mathcal{Z}$$

and

$$h(x) := g(f(x)), \ x \in \mathcal{E}$$

then h *is continuous at a point* $p \in \mathcal{E}$ *if* f *is continuous at* p *and* g *is continuous at* $f(p)$.

2. *If* $f : \mathcal{X} \to \mathbb{R}^n$ *and* $f(x) := (f_1(x), ..., f_n(x))$ *then* f *is continuous if and only if all* $f_i(x)$ $\left(i = \overline{1, n}\right)$ *are continuous.*

3. *If* $f, g : \mathcal{X} \to \mathbb{R}^n$ *are continuos mappings then* $f + g$ *and* (f, g) *are continuous too on* $\mathcal{X}$.

4. *A mapping* $f : \mathcal{X} \to \mathcal{Y}$ *is continuous on* $\mathcal{X}$ *if and only if* $f^{-1}(\mathcal{V})$ *is open (closed) in* $\mathcal{X}$ *for every open (closed) set* $\mathcal{V} \subset \mathcal{Y}$.

Continuity, compactness and connectedness

Theorem 14.11 *If* $f : \mathcal{X} \to \mathcal{Y}$ *is a continuous mapping of a compact metric space* $\mathcal{X}$ *into a metric space* $\mathcal{Y}$ *then* $f(\mathcal{X})$ *is compact.*

Proof. Let $\{\mathcal{V}_\alpha\}$ be an open cover of $f(\mathcal{X})$. By continuity of f and in view of Proposition 14.5 it follows that each of the sets $f^{-1}(\mathcal{V}_\alpha)$ is open. By the compactness of $\mathcal{X}$ there are finitely many indices $\alpha_1, ..., \alpha_n$ such that

$$\mathcal{X} \subset \bigcup_{i=1}^{n} f^{-1}(\mathcal{V}_{\alpha_i}) \tag{14.53}$$

Since $f(f^{-1}(\mathcal{E})) \subset \mathcal{E}$ for any $\mathcal{E} \subset \mathcal{Y}$ it follows that (14.53) implies that $f(\mathcal{X}) \subset \bigcup_{\alpha=1}^{n} \mathcal{V}_{\alpha_i}$. This completes the proof. ■

Corollary 14.2 *If* $f : \mathcal{X} \to \mathbb{R}^n$ *is a continuous mapping of a compact metric space* $\mathcal{X}$ *into* $\mathbb{R}^n$ *then* $f(\mathcal{X})$ *is closed and bounded, that is, it contains its all limit points and* $\|f(x)\| \leq M < \infty$ *for any* $x \in \mathcal{X}$.

Proof. It follows directly from Theorems 14.11 and 14.4. ■

The next theorem is particular important when f is real.

Theorem 14.12 (Weierstrass theorem) *If* $f : \mathcal{X} \to \mathbb{R}^n$ *is a* ***continuous mapping*** *of a* ***compact metric space*** $\mathcal{X}$ *into* $\mathbb{R}$ *and*

$$M = \sup_{x \in \mathcal{X}} f(x), \ m = \inf_{x \in \mathcal{X}} f(x)$$

then there exist points $x_M, x_m \in \mathcal{X}$ *such that*

$$M = f(x_M), \ m = f(x_m)$$

This means that f attains its maximum (at x_M) and its minimum (at x_m), that is,

$$\boxed{M = \sup_{x\in\mathcal{X}} f(x) = \max_{x\in\mathcal{X}} f(x),\ m = \inf_{x\in\mathcal{X}} f(x) = \min_{x\in\mathcal{X}} f(x)}$$

Proof. By Theorem 14.11 and its Corollary it follows that $f(\mathcal{X})$ is closed and bounded set (say, $\mathcal{E}$) of real numbers. So, if $M \in \mathcal{E}$ then $M \in \text{cl}\mathcal{E}$. Suppose $M \notin \mathcal{E}$. Then for any $\varepsilon > 0$ there is a point $y \in \mathcal{E}$ such that $M - \varepsilon < y < M$, for otherwise $(M - \varepsilon)$ would be an upper bound. Thus y is a limit point of $\mathcal{E}$. Hence, $y \in \text{cl}\mathcal{E}$ that proves the theorem. ■

The next theorem deals with the continuity property for inverse continuous one-to-one mappings.

Theorem 14.13 *If $f : \mathcal{X} \to \mathcal{Y}$ is a continuous one-to-one mapping of a compact metric space $\mathcal{X}$ into a metric space $\mathcal{Y}$ then the inverse mapping $f^{-1} : \mathcal{Y} \to \mathcal{X}$ defined by*

$$f^{-1}(f(x)) = x \in \mathcal{X}$$

is a continuous mapping too.

Proof. By Proposition 14.4, applied to f^{-1} instead of f, one can see that it is sufficient to prove that $f(\mathcal{V})$ is an open set of $\mathcal{Y}$ for any open set $\mathcal{V} \subset \mathcal{X}$. Fixing a set $\mathcal{V}$ we may conclude that the complement $\mathcal{V}^c$ of $\mathcal{V}$ is closed in $\mathcal{X}$ and, hence, by Proposition 14.5 it is a compact. As the result, $f(\mathcal{V}^c)$ is a compact subset of $\mathcal{Y}$ (14.11) and so, by Theorem 14.2, it is closed in $\mathcal{Y}$. Since f is one-to-one and onto, $f(\mathcal{V})$ is the compliment of $f(\mathcal{V}^c)$ and hence, it is open. This completes the proof. ■

Uniform continuity

Definition 14.17 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping of a space $\mathcal{X}$ into a metric space $\mathcal{Y}$. A mapping f is said to be*

a) ***uniformly continuous*** *on $\mathcal{X}$ if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d_{\mathcal{Y}}(f(x), f(x')) < \varepsilon$ for all $x, x' \in \mathcal{X}$ for which $d_{\mathcal{X}}(x, x') < \delta$.*

b) ***uniformly Lipschitz continuous*** *on a (x, z)-set $\mathcal{E}$ with respect to x, if there exists a positive constant $L_f < \infty$ such that*

$$d_{\mathcal{Y}}\left(f\left(x,z\right), f\left(x',z\right)\right) \leq L_f d_{\mathcal{X}}\left(x,x'\right)$$

for all $x, x', z \in \mathcal{E}$.

Remark 14.4 *The different between the concepts of* ***continuity*** *and* ***uniform continuity*** *concerns two aspects:*

a) *uniform continuity is a property of a function on a set, whereas continuity is defined for a function in a single point;*

b) *δ, participating in the definition (14.50) of continuity, is a function of ε and a point p, that is, $\delta = \delta\left(\varepsilon, p\right)$, whereas δ, participating in the definition (14.17) of simple continuity, is a function of ε only serving for all points of a set (space) $\mathcal{X}$, that is, $\delta = \delta\left(\varepsilon\right)$.*

Evidently that any uniformly continues function is continuous but not inverse. The next theorem shows when both concepts coincide.

Theorem 14.14 *If $f : \mathcal{X} \to \mathcal{Y}$ is a continuous mapping of a compact metric space $\mathcal{X}$ into a metric space $\mathcal{Y}$ then f is uniformly continuous on $\mathcal{X}$.*

Proof. Continuity means that for any point $p \in \mathcal{X}$ and any $\varepsilon > 0$ we can associate a number $\delta\left(\varepsilon, p\right)$ such that

$$x \in \mathcal{X},\ d_{\mathcal{X}}\left(x,p\right) < \delta\left(\varepsilon,p\right) \text{ implies } d_{\mathcal{Y}}\left(f\left(x\right), f\left(p\right)\right) < \varepsilon/2 \quad (14.54)$$

Define the set

$$\mathcal{J}\left(p\right) := \left\{x \in \mathcal{X} : d_{\mathcal{X}}\left(x,p\right) < \delta\left(\varepsilon,p\right)/2\right\}$$

Since $p \in \mathcal{J}\left(p\right)$ the collection of all sets $\mathcal{J}\left(p\right)$ is an open cover of $\mathcal{X}$ and by the compactness of $\mathcal{X}$ there are a finite set of points $p_1, ..., p_n$ such that

$$\mathcal{X} \subset \mathcal{J}\left(p_1\right) \cup \cdots \cup \mathcal{J}\left(p_n\right) \quad (14.55)$$

Put

$$\tilde{\delta}\left(\varepsilon\right) := \frac{1}{2}\min\left\{\delta\left(\varepsilon,p_1\right), ..., \delta\left(\varepsilon,p_n\right)\right\} > 0$$

Now let $x \in \mathcal{X}$ satisfies the inequality $d_{\mathcal{X}}(x,p) < \tilde{\delta}(\varepsilon)$. By the compactness (namely, by (14.55)) there is an integer m $(1 \leq m \leq n)$ such that $p \in \mathcal{J}(p_m)$ that implies

$$d_{\mathcal{X}}(x,p_m) < \frac{1}{2}\delta(\varepsilon,p_m)$$

and, as the result,

$$d_{\mathcal{X}}(x,p_m) \leq d_{\mathcal{X}}(x,p) + d_{\mathcal{X}}(p,p_m) \leq \tilde{\delta}(\varepsilon) + \frac{1}{2}\delta(\varepsilon,p_m) \leq \delta(\varepsilon,p_m)$$

Finally, by (14.54)

$$d_{\mathcal{Y}}(f(x), f(p)) \leq d_{\mathcal{Y}}(f(x), f(p_m)) + d_{\mathcal{Y}}(f(p_m), f(p)) \leq \varepsilon$$

that completes the proof. ■

Remark 14.5 *The alternative proof of this theorem may be obtained in the following manner: assuming that f is not uniformly continuous we conclude that there exists $\varepsilon > 0$ and the sequences $\{x_n\}$, $\{p_n\}$ on $\mathcal{X}$ such that $d_{\mathcal{X}}(x_n,p_n) \underset{n\to\infty}{\to} 0$ but $d_{\mathcal{Y}}(f(x_n), f(p_n)) > \varepsilon$. The last is in a contradiction with Theorem 14.3.*

Next examples shows that compactness is essential in the hypotheses of the previous theorems.

Example 14.6 *If $\mathcal{E}$ is a non compact in $\mathbb{R}$ then*

1. *There is a continuous function on $\mathcal{E}$ which is not bounded, for example,*

$$f(x) = \frac{1}{x-1}, \ \mathcal{E} := \{x \in \mathbb{R} : |x| < 1\}$$

   Here, $\mathcal{E}$ is a non compact, $f(x)$ is continuous on $\mathcal{E}$, but evidently unbounded. It is easy to check that it is not uniformly continuous.

2. *There exists a continuous and bounded function on $\mathcal{E}$ which has no maximum, for example,*

$$f(x) = \frac{1}{1+(x-1)^2}, \ \mathcal{E} := \{x \in \mathbb{R} : |x| < 1\}$$

Evidently that

$$\sup_{x \in \mathcal{E}} f(x) = 1$$

whereas $\frac{1}{2} \leq f(x) < 1$ *and, hence, has no maximum on* $\mathcal{E}$*.*

Continuity of a family of functions: equicontinuity

Definition 14.18 *A family* $\boldsymbol{F}$ *of functions* $f(x)$ *defined on some* x*-set* $\mathcal{E}$ *is said to be* ***equicontinuous*** *if for any* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon)$*, the same for all class* $\boldsymbol{F}$*, such that* $d_{\mathcal{X}}(x, y) < \delta$ *implies* $d_{\mathcal{Y}}(f(x), f(y)) < \varepsilon$ *for all* $x, y \in \mathcal{E}$ *and any* $f \in \boldsymbol{F}$*.*

The most frequently encountered equicontinuous families $\boldsymbol{F}$ occur when $f \in \boldsymbol{F}$ are uniformly Lipschitz continuous on $\mathcal{X} \subseteq \mathbb{R}^n$ and there exists a $L_f > 0$ which is a Lipschitz constant for all $f \in \boldsymbol{F}$. In this case $\delta = \delta(\varepsilon)$ can be chosen as $\delta = \varepsilon / L_f$.

The following claim can be easily proven.

Claim 14.5 *If a sequence of continuous functions on a compact set* $\mathcal{X} \subseteq \mathbb{R}^n$ *is uniformly convergent on* $\mathcal{X}$*, then it is uniformly bounded and equicontinuous.*

The next two assertions usually referred to as the *Ascoli-Arzelà's theorems(see the reference in (Hartman 2002)).* They will be used below for the analysis of Ordinary Differential Equations.

Theorem 14.15 (on the propagation, Ascoli-Arzelà 1883-1895) *Let on a compact* x*-set of* $\mathcal{E}$*, the sequence of functions* $\{f_n(x)\}_{n=1,2,\ldots}$ *be equicontinuous and convergent on a dense subset of* $\mathcal{E}$*. Then there exists a subsequence* $\{f_{n_k}(x)\}_{k=1,2,\ldots}$ *which is uniformly convergent on* $\mathcal{E}$*.*

Another version of the same fact looks as follows.

Theorem 14.16 (on the selection, Ascoli-Arzelà 1883-1895) *Let on a compact* x*-set of* $\mathcal{E} \subset \mathbb{R}^n$*, the sequence of functions* $\{f_n(x)\}_{n=1,2,\ldots}$ *be uniformly bounded and equicontinuous. Then there exists a subsequence* $\{f_{n_k}(x)\}_{k=1,2,\ldots}$ *which is uniformly convergent on* $\mathcal{E}$*.*

Proof. Let us consider the set of all rational numbers $\boldsymbol{R} \subseteq \mathcal{E}$. Since $\boldsymbol{R}$ is countable, all of its elements can be designated by numbers, i.e., $\boldsymbol{R} = \{r_j\}$ $(j = 1, ...)$. The numerical vector-sequence $\{f_n(r_1)\}_{n=1,2,...}$ is norm-bounded, say, $\|f_n(r_1)\| \leq M$. Hence, we can choose a convergent sequence $\{f_{n_k}(r_2)\}_{k=1,2,...}$ which is also bounded by the same M. Continuing this process we obtain a subsequence $\{f_p(r_q)\}_{p=1,2,...}$ that converges in a point r_q, $q = 1, 2, ...$ Let $f_p := f_p(r_p)$. Show that the sequence $\{f_p\}$ is uniformly convergent on $\mathcal{E}$ to a continuous function $f \in C(\mathcal{E})$. In fact, $\{f_p\}$ converges in any point of $\boldsymbol{R}$ by the construction. To establish it convergence in any point of $\mathcal{E}$, it is sufficient to to show that for any fixed $x \in \mathcal{E}$ the sequence $\{f_p(x)\}$ converges in itself. Since $\{f_p(x)\}$ is equicontinuous, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for $\|x - x'\| < \delta$ and $x, x' \in \mathcal{E}$ there is $\|f_p(x) - f_p(x')\| < \varepsilon$. Choose r_j such that $\|x - r_j\| < \delta$ that implies $\|f_p(x) - f_p(r_j)\| < \varepsilon$. But the sequence $\{f_p(r_j)\}$ converges in itself. Hence, there is a number p_0 such that $\|f_p(x) - f_{p'}(x')\| < \varepsilon$ whenever $p, p' > p_0$. So,

$$\|f_p(x) - f_{p'}(x')\| \leq \|f_p(x) - f_p(r_j)\| + \\ \|f_p(r_j) - f_{p'}(r_j)\| + \|f_{p'}(r_j) - f_{p'}(x')\| \leq 3\varepsilon$$

Thus $\{f_p(x)\}$ converges at each $x \in \mathcal{E}$. It remains to prove that $\{f_p(x)\}$ converges uniformly on $\mathcal{E}$ and, therefore, its limit f is from $C(\mathcal{E})$. Again, by the assumption on equicontinuity, one can cover the set $\mathcal{E}$ with the finite δ-set containing, say, l subsets. In each of them select a rational numbers, say, $r_1, ..., r_l$. By the convergence of $\{f_p(x)\}$ there exists p_0 such that $\|f_p(r_j) - f_{p'}(r_j)\| < \varepsilon$ whenever $p, p' > p_0$, so that

$$\|f_p(x) - f_{p'}(x)\| \leq \|f_p(x) - f_p(r_j)\| + \\ \|f_p(r_j) - f_{p'}(r_j)\| + \|f_{p'}(r_j) - f_{p'}(x)\| \leq 3\varepsilon$$

where j is selected in such a way that r_j belongs to the same δ-subset as x. Taking $p' \to \infty$, this inequality implies $\|f_p(x) - f(x)\| \leq 3\varepsilon$ for all x from the considered δ-subset. but this exactly means the uniform converges of $\{f_p(x)\}$. Theorem is proven. ■

Connectedness

The definition of the connectedness of a set $\mathcal{E}$ has been given in Definition 14.7. Here we will discuss its relation with the continuity property of a function f.

Lemma 14.2 *If $f : \mathcal{X} \to \mathcal{Y}$ is a continuous mapping of a metric space $\mathcal{X}$ into a metric space $\mathcal{Y}$, and if $\mathcal{E}$ is a connected subset of $\mathcal{X}$, then $f(\mathcal{E})$ is connected.*

Proof. On the contrary, assume that $f(\mathcal{E}) = \mathcal{A} \cup \mathcal{B}$ with non empty sets $\mathcal{A}, \mathcal{B} \subset \mathcal{Y}$ such that $\mathcal{A} \cap \mathcal{B} = \varnothing$. Put $\mathcal{G} = \mathcal{E} \cap f^{-1}(\mathcal{A})$ and $\mathcal{H} = \mathcal{E} \cap f^{-1}(\mathcal{B})$. Then $\mathcal{E} = \mathcal{G} \cup \mathcal{H}$ and both $\mathcal{G}$ and $\mathcal{H}$ are non empty. Since $\mathcal{A} \subset \mathrm{cl}\mathcal{A}$ it follows that $\mathcal{G} \subset f^{-1}(\mathrm{cl}\mathcal{A})$ and $f(\mathrm{cl}\mathcal{G}) \subset \mathrm{cl}\mathcal{A}$. Taking into account that $f(\mathcal{H}) = \mathcal{B}$ and $\mathrm{cl}\mathcal{A} \cap \mathcal{B} = \varnothing$ we may conclude that $\mathcal{G} \cap \mathcal{H} = \varnothing$. By the same argument we conclude that $\mathcal{G} \cap \mathrm{cl}\mathcal{H} = \varnothing$. Thus, $\mathcal{G}$ and $\mathcal{H}$ are separated that is impossible if $\mathcal{E}$ is connected. Lemma is proven. ■

This theorem serves as an instrument to state the important result in $\mathbb{R}$ which is known as the Bolzano theorem which concerns a global property of real-valued functions continuous on a compact interval $[a, b] \in \mathbb{R}$: if $f(a) < 0$ and $f(b) > 0$ then the graph of the function $f(x)$ must cross the x - axis somewhere in between. But this theorem as well as other results, concerning the analysis of functions given on $\mathbb{R}^n$, will be considered in details below in the chapter named "*Elements of Real Analyses*".

Homeomorphisms

Definition 14.19 *Let $f : \mathcal{S} \to \mathcal{T}$ be a function mapping points from one metric space $(\mathcal{S}, d_{\mathcal{S}})$ to another $(\mathcal{T}, d_{\mathcal{T}})$ such that it is one-to-one mapping or, in other words, $f^{-1} : \mathcal{T} \to \mathcal{S}$ exists. If additionally f is continuous on $\mathcal{S}$ and f^{-1} on $\mathcal{T}$ then such mapping f is called a* ***topological mapping*** *or* ***homeomorphism*** *, and the spaces $(\mathcal{S}, d_{\mathcal{S}})$ and $(f(\mathcal{S}), d_{\mathcal{T}})$ are said to be* ***homeomorphic****.*

It is clear from this definition that if f is homeomorphism then f^{-1} is homeomorphism too. The important particular case of a homeomorphism is, the so-called, an *isometry*, i.e., it is a one-to one continuous mapping which preserves the metric, namely, which for all $x, x' \in \mathcal{S}$ keeps the identity

$$\boxed{d_{\mathcal{T}}(f(x), f(x')) = d_{\mathcal{S}}(x, x')} \tag{14.56}$$

14.2.6 The contraction principle and a fixed point theorem

Definition 14.20 *Let $\mathcal{X}$ be a metric space with a metric d. If φ maps $\mathcal{X}$ into $\mathcal{X}$ and if there is a number $c \in [0,1)$ such that*

$$\boxed{d(\varphi(x), \varphi(x')) \le cd(x, x')} \tag{14.57}$$

*for all $x, x' \in \mathcal{X}$, then φ is said to be a **contraction** of $\mathcal{X}$ into $\mathcal{X}$.*

Theorem 14.17 (the fixed point theorem) *If $\mathcal{X}$ is a complete metric space and if φ is a contraction of $\mathcal{X}$ into $\mathcal{X}$, then there exists one and only one point $x \in \mathcal{X}$ such that*

$$\boxed{\varphi(x) = x} \tag{14.58}$$

Proof. Pick $x_0 \in \mathcal{X}$ arbitrarily and define the sequence $\{x_n\}$ recursively by setting $x_{n+1} = \varphi(x_n)$, $n = 0, 1, ...$ Then, since φ is a contraction, we have

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \le \\ cd(x_n, x_{n-1}) \le \cdots \le c^n d(x_1, x_0)$$

Taking $m > n$ and in view of the triangle inequality, it follows

$$d(x_m, x_n) \le \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \le \left(c^{m-1} + \cdots + c^n\right) d(x_1, x_0) \le \\ c^n \left(c^{m-1-n} + \cdots + 1\right) d(x_1, x_0) \le c^n (1-c)^{-1} d(x_1, x_0)$$

Thus $\{x_n\}$ is a Cauchy sequence, and since $\mathcal{X}$ is a complete metric space, it should converge, that is, there exists $\lim_{n\to\infty} x_n := x$. And, since φ is a contraction, it is continuous (in fact, uniformly continuous). Therefore $\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_n = x$. The uniqueness follows from the following consideration. Assume that there exists another point $y \in \mathcal{X}$ such that $\varphi(y) = y$. Then by (14.57) it follows $d(x, y) \le cd(\varphi(x), \varphi(y)) = cd(x, y)$ which may only happen if $d(x, y) = 0$ that proves the theorem. ■

14.3 Resume

The properties of sets which remain invariant under every topological mapping is usually called the *topological properties*. Thus properties of being open, closed, or compact are topological properties.