Chapter 3

Fundamental Properties

This chapter states some fundamental properties of the solutions of ordinary differential equations, like existence, uniqueness, continuous dependence on initial conditions, and continuous dependence on parameters. These properties are essential for the state equation $\dot{x} = f(t, x)$ to be a useful mathematical model of a physical system. In experimenting with a physical system such as the pendulum, we expect that starting the experiment from a given initial state at time t_0, the system will move and its state will be defined in the (at least immediate) future time $t > t_0$. Moreover, with a deterministic system, we expect that if we could repeat the experiment exactly, we would get exactly the same motion and the same state at $t > t_0$. For the mathematical model to predict the future state of the system from its current state at t_0, the initial-value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{3.1}$$

must have a unique solution. This is the question of existence and uniqueness that is addressed in Section 3.1. It is shown that existence and uniqueness can be ensured by imposing some constraints on the right-hand side function $f(t, x)$. The key constraint used in Section 3.1 is the Lipschitz condition, whereby $f(t, x)$ satisfies the inequality[1]

$$\|f(t, x) - f(t, y)\| \le L\|x - y\| \tag{3.2}$$

for all (t, x) and (t, y) in some neighborhood of (t_0, x_0).

An essential factor in the validity of any mathematical model is the continuous dependence of its solutions on the data of the problem. The least we should expect from a mathematical model is that arbitrarily small errors in the data will not result in large errors in the solutions obtained by the model. The data of the initial-value problem (3.1) are the initial state x_0, the initial time t_0, and the right-hand side

[1] $\|\cdot\|$ denotes any p-norm, as defined in Appendix A.

function $f(t, x)$. Continuous dependence on the initial conditions (t_0, x_0) and on the parameters of f are studied in Section 3.2. If f is differentiable with respect to its parameters, then the solution will be differentiable with respect to these parameters. This is shown in Section 3.3 and is used to derive sensitivity equations that describe the effect of small parameter variations on the performance of the system. The continuity and differentiability results of Sections 3.2 and 3.3 are valid only on finite time intervals. Continuity results on the infinite time interval will be given later, after stability concepts have been introduced.[2]

The chapter ends with a brief statement of a comparison principle that bounds the solution of a scalar differential inequality $\dot{v} \leq f(t, v)$ by the solution of the differential equation $\dot{u} = f(t, u)$.

3.1 Existence and Uniqueness

In this section, we derive sufficient conditions for the existence and uniqueness of the solution of the initial-value problem (3.1). By a solution of (3.1) over an interval $[t_0, t_1]$, we mean a continuous function $x : [t_0, t_1] \to R^n$ such that $\dot{x}(t)$ is defined and $\dot{x}(t) = f(t, x(t))$ for all $t \in [t_0, t_1]$. If $f(t, x)$ is continuous in t and x, then the solution $x(t)$ will be continuously differentiable. We will assume that $f(t, x)$ is continuous in x, but only piecewise continuous in t, in which case, a solution $x(t)$ could only be piecewise continuously differentiable. The assumption that $f(t, x)$ be piecewise continuous in t allows us to include the case when $f(t, x)$ depends on a time-varying input that may experience step changes with time.

A differential equation with a given initial condition might have several solutions. For example, the scalar equation

$$\dot{x} = x^{1/3}, \quad \text{with} \quad x(0) = 0 \tag{3.3}$$

has a solution $x(t) = (2t/3)^{3/2}$. This solution is not unique, since $x(t) \equiv 0$ is another solution. In noting that the right-hand side of (3.3) is continuous in x, it is clear that continuity of $f(t, x)$ in its arguments is not sufficient to ensure uniqueness of the solution. Extra conditions must be imposed on the function f. The question of existence of a solution is less stringent. In fact, continuity of $f(t, x)$ in its arguments ensures that there is at least one solution. We will not prove this fact here.[3] Instead, we prove an easier theorem that employs the Lipschitz condition to show existence and uniqueness.

Theorem 3.1 (Local Existence and Uniqueness) *Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

[2] See, in particular, Section 9.4.

[3] See [135, Theorem 2.3] for a proof.

$\forall\ x, y \in B = \{x \in R^n \mid \|x - x_0\| \le r\}$, $\forall\ t \in [t_0, t_1]$. *Then, there exists some* $\delta > 0$ *such that the state equation* $\dot{x} = f(t, x)$ *with* $x(t_0) = x_0$ *has a unique solution over* $[t_0, t_0 + \delta]$. ◇

Proof: See Appendix C.1.

The key assumption in Theorem 3.1 is the Lipschitz condition (3.2). A function satisfying (3.2) is said to be *Lipschitz* in x, and the positive constant L is called a *Lipschitz constant.* We also use the words *locally Lipschitz* and *globally Lipschitz* to indicate the domain over which the Lipschitz condition holds. Let us introduce the terminology first for the case when f depends only on x. A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset R^n$ if each point of D has a neighborhood D_0 such that f satisfies the Lipschitz condition (3.2) for all points in D_0 with some Lipschitz constant L_0. We say that f is Lipschitz on a set W if it satisfies (3.2) for all points in W, with the same Lipschitz constant L. A locally Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D. However, a locally Lipschitz function on a domain D is Lipschitz on every compact (closed and bounded) subset of D (Exercise 3.19). A function $f(x)$ is said to be *globally Lipschitz* if it is Lipschitz on R^n. The same terminology is extended to a function $f(t, x)$, provided the Lipschitz condition holds uniformly in t for all t in a given interval of time. For example, $f(t, x)$ is locally Lipschitz in x on $[a, b] \times D \subset R \times R^n$ if each point $x \in D$ has a neighborhood D_0 such that f satisfies (3.2) on $[a, b] \times D_0$ with some Lipschitz constant L_0. We say that $f(t, x)$ is locally Lipschitz in x on $[t_0, \infty) \times D$ if it is locally Lipschitz in x on $[a, b] \times D$ for every compact interval $[a, b] \subset [t_0, \infty)$. A function $f(t, x)$ is Lipschitz in x on $[a, b] \times W$ if it satisfies (3.2) for all $t \in [a, b]$ and all points in W, with the same Lipschitz constant L.

When $f : R \to R$, the Lipschitz condition can be written as

$$\frac{|f(y) - f(x)|}{|y - x|} \le L$$

which implies that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L. Therefore, any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point. For example, any discontinuous function is not locally Lipschitz at the point of discontinuity. As another example, the function $f(x) = x^{1/3}$, which was used in (3.3), is not locally Lipschitz at $x = 0$ since $f'(x) = (1/3)x^{-2/3} \to \infty$ as $x \to 0$. On the other hand, if $|f'(x)|$ is bounded by a constant k over the interval of interest, then $f(x)$ is Lipschitz on the same interval with Lipschitz constant $L = k$. This observation extends to vector-valued functions, as demonstrated by Lemma 3.1.

Lemma 3.1 *Let* $f : [a, b] \times D \to R^m$ *be continuous for some domain* $D \subset R^n$. *Suppose that* $[\partial f / \partial x]$ *exists and is continuous on* $[a, b] \times D$. *If, for a convex subset*

$W \subset D$, there is a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L$$

on $[a,b] \times W$, then

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all $t \in [a,b]$, $x \in W$, and $y \in W$. ◇

Proof: Let $\|\cdot\|_p$ be the underlying norm for any $p \in [1,\infty]$, and determine $q \in [1,\infty]$ from the relationship $1/p + 1/q = 1$. Fix $t \in [a,b]$, $x \in W$, and $y \in W$. Define $\gamma(s) = (1-s)x + sy$ for all $s \in R$ such that $\gamma(s) \in D$. Since $W \subset D$ is convex, $\gamma(s) \in W$ for $0 \leq s \leq 1$. Take $z \in R^m$ such that[4]

$$\|z\|_q = 1 \quad \text{and} \quad z^T[f(t,y) - f(t,x)] = \|f(t,y) - f(t,x)\|_p$$

Set $g(s) = z^T f(t,\gamma(s))$. Since $g(s)$ is a real-valued function, which is continuously differentiable in an open interval that includes $[0,1]$, we conclude by the mean value theorem that there is $s_1 \in (0,1)$ such that

$$g(1) - g(0) = g'(s_1)$$

Evaluating g at $s = 0$, $s = 1$, and calculating $g'(s)$ by using the chain rule, we obtain

$$\begin{aligned} z^T[f(t,y) - f(t,x)] &= z^T \frac{\partial f}{\partial x}(t,\gamma(s_1))(y-x) \\ \|f(t,y) - f(t,x)\|_p &\leq \|z\|_q \left\| \frac{\partial f}{\partial x}(t,\gamma(s_1)) \right\|_p \|y-x\|_p \leq L\|y-x\|_p \end{aligned}$$

where we used the Hölder inequality $|z^T w| \leq \|z\|_q \|w\|_p$. □

The lemma shows how a Lipschitz constant can be calculated using knowledge of $[\partial f/\partial x]$.

The Lipschitz property of a function is stronger than continuity. It can be easily seen that if $f(x)$ is Lipschitz on W, then it is uniformly continuous on W (Exercise 3.20). The converse is not true, as seen from the function $f(x) = x^{1/3}$, which is continuous, but not locally Lipschitz at $x = 0$. The Lipschitz property is weaker than continuous differentiability, as stated in the next lemma.

Lemma 3.2 *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times D$, for some domain $D \subset R^n$, then f is locally Lipschitz in x on $[a,b] \times D$.* ◇

[4]Such z always exists. (See Exercise 3.21.)

Proof: For $x_0 \in D$, let r be so small that the ball $D_0 = \{x \in R^n \mid \|x - x_0\| \le r\}$ is contained in D. The set D_0 is convex and compact. By continuity, $[\partial f/\partial x]$ is bounded on $[a,b] \times D_0$. Let L_0 be a bound for $\|\partial f/\partial x\|$ on $[a,b] \times D_0$. By Lemma 3.1. f is Lipschitz on $[a,b] \times D_0$ with Lipschitz constant L_0. □

It is left to the reader (Exercise 3.22) to extend the proof of Lemma 3.1 to prove the next lemma.

Lemma 3.3 *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lipschitz in x on $[a,b] \times R^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b] \times R^n$.* ◇

Example 3.1 The function

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

is continuously differentiable on R^2. Hence, it is locally Lipschitz on R^2. It is not globally Lipschitz since $[\partial f/\partial x]$ is not uniformly bounded on R^2. On any compact subset of R^2. f is Lipschitz. Suppose that we are interested in calculating a Lipschitz constant over the convex set $W = \{x \in R^2 \mid |x_1| \le a_1,\ |x_2| \le a_2\}$. The Jacobian matrix is given by

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$$

Using $\|.\|_\infty$ for vectors in R^2 and the induced matrix norm for matrices, we have

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\{|-1 + x_2| + |x_1|,\ |x_2| + |1 - x_1|\}$$

All points in W satisfy

$$|-1 + x_2| + |x_1| \le 1 + a_2 + a_1 \quad \text{and} \quad |x_2| + |1 - x_1| \le a_2 + 1 + a_1$$

Hence,

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty \le 1 + a_1 + a_2$$

and a Lipschitz constant can be taken as $L = 1 + a_1 + a_2$. △

Example 3.2 The function

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

is not continuously differentiable on R^2. Let us check its Lipschitz property by examining $f(x) - f(y)$. Using $\|.\|_2$ for vectors in R^2 and the fact that the saturation function $\text{sat}(\cdot)$ satisfies

$$|\text{sat}(\eta) - \text{sat}(\xi)| \le |\eta - \xi|$$

we obtain

$$\begin{aligned} \|f(x) - f(y)\|_2^2 &\leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned}$$

Using the inequality

$$a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right\} \times \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2^2$$

we conclude that

$$\|f(x) - f(y)\|_2 \leq \sqrt{2.618}\ \|x - y\|_2, \quad \forall\ x, y \in R^2$$

Here we have used a property of positive semidefinite symmetric matrices; that is, $x^T P x \leq \lambda_{\max}(P)\ x^T x$, for all $x \in R^n$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the matrix. A more conservative (larger) Lipschitz constant will be obtained if we use the more conservative inequality

$$a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)$$

resulting in a Lipschitz constant $L = \sqrt{3}$. △

In these two examples, we have used $\|\cdot\|_\infty$ in one case and $\|\cdot\|_2$ in the other. Due to equivalence of norms, the choice of a norm on R^n does not affect the Lipschitz property of a function. It only affects the value of the Lipschitz constant (Exercise 3.5). Example 3.2 illustrates the fact that the Lipschitz condition (3.2) does not uniquely define the Lipschitz constant L. If (3.2) is satisfied with some positive constant L, it is satisfied with any constant larger than L. This nonuniqueness can be removed by defining L to be the smallest constant for which (3.2) is satisfied, but we seldom need to do that.

Theorem 3.1 is a local theorem since it guarantees existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where δ may be very small. In other words, we have no control on δ; hence, we cannot ensure existence and uniqueness over a given time interval $[t_0, t_1]$. However, one may try to extend the interval of existence by repeated applications of the local theorem. Starting at a time t_0, with an initial state $x(t_0) = x_0$, Theorem 3.1 shows that there is a positive constant δ (dependent on x_0) such that the state equation (3.1) has a unique solution over the time interval $[t_0, t_0 + \delta]$. Now, taking $t_0 + \delta$ as a new initial time and $x(t_0 + \delta)$ as a new initial state, one may try to apply Theorem 3.1 to establish existence of the solution beyond $t_0 + \delta$. If the conditions of the theorem are satisfied at $(t_0 + \delta, x(t_0 + \delta))$, then there exists $\delta_2 > 0$ such that the equation has a unique solution over $[t_0 + \delta, t_0 + \delta + \delta_2]$ that passes through the point $(t_0 + \delta, x(t_0 + \delta))$. We piece together the solutions over $[t_0, t_0 + \delta]$ and $[t_0 + \delta, t_0 + \delta + \delta_2]$ to establish the existence of a unique solution over $[t_0, t_0 + \delta + \delta_2]$. This idea can be repeated to keep extending the solution. However, in general, the interval of existence of the solution cannot be extended indefinitely because the conditions of Theorem 3.1 may cease to hold. There is a maximum interval $[t_0, T)$ where the unique solution starting at (t_0, x_0) exists.[5] In general, T may be less than t_1, in which case as $t \to T$, the solution leaves any compact set over which f is locally Lipschitz in x (Exercise 3.26).

Example 3.3 Consider the scalar system

$$\dot{x} = -x^2, \quad \text{with } x(0) = -1$$

The function $f(x) = -x^2$ is locally Lipschitz for all $x \in R$. Hence, it is Lipschitz on any compact subset of R. The unique solution

$$x(t) = \frac{1}{t-1}$$

exists over $[0, 1)$. As $t \to 1$, $x(t)$ leaves any compact set. △

The phrase "finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time. In Example 3.3, we say that the trajectory has a finite escape time at $t = 1$.

In view of the discussion preceding Example 3.3, one may pose the following question: When is it guaranteed that the solution can be extended indefinitely? One way to answer the question is to require additional conditions which ensure that the solution $x(t)$ will always be in a set where $f(t, x)$ is uniformly Lipschitz in x. This is done in the next theorem by requiring f to satisfy a global Lipschitz condition. The theorem establishes the existence of a unique solution over $[t_0, t_1]$, where t_1 may be arbitrarily large.

Theorem 3.2 (Global Existence and Uniqueness) *Suppose that $f(t, x)$ is piecewise continuous in t and satisfies*

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|$$

$\forall\ x, y \in R^n$, $\forall\ t \in [t_0, t_1]$. *Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.* ◇

Proof: See Appendix C.1.

Example 3.4 Consider the linear system

$$\dot{x} = A(t)x + g(t) = f(t, x)$$

where $A(t)$ and $g(t)$ are piecewise continuous functions of t. Over any finite interval of time $[t_0, t_1]$, the elements of $A(t)$ are bounded. Hence, $\|A(t)\| \le a$, where $\|A\|$ is any induced matrix norm. The conditions of Theorem 3.2 are satisfied since

$$\|f(t, x) - f(t, y)\| = \|A(t)(x - y)\| \le \|A(t)\|\ \|x - y\| \le a\|x - y\|$$

[5] For a proof of this statement, see [81, Section 8.5] or [135, Section 2.3].

for all $x, y \in R^n$ and $t \in [t_0, t_1]$. Therefore, Theorem 3.2 shows that the linear system has a unique solution over $[t_0, t_1]$. Since t_1 can be arbitrarily large, we can also conclude that if $A(t)$ and $g(t)$ are piecewise continuous $\forall\, t \geq t_0$, then the system has a unique solution $\forall\, t \geq t_0$. Hence, the system cannot have a finite escape time. △

For the linear system of Example 3.4, the global Lipschitz condition of Theorem 3.2 is a reasonable requirement. This may not be the case for nonlinear systems, in general. We should distinguish between the local Lipschitz requirement of Theorem 3.1 and the global Lipschitz requirement of Theorem 3.2. Local Lipschitz property of a function is basically a smoothness requirement. It is implied by continuous differentiability. Except for discontinuous nonlinearities, which are idealizations of physical phenomena, it is reasonable to expect models of physical systems to have locally Lipschitz right-hand side functions. Examples of continuous functions that are not locally Lipschitz are quite exceptional and rarely arise in practice. The global Lipschitz property, on the other hand, is restrictive. Models of many physical systems fail to satisfy it. One can easily construct smooth meaningful examples that do not have the global Lipschitz property, but do have unique global solutions, which is an indication of the conservative nature of Theorem 3.2.

Example 3.5 Consider the scalar system

$$\dot{x} = -x^3 = f(x)$$

The function $f(x)$ does not satisfy a global Lipschitz condition since the Jacobian $\partial f/\partial x = -3x^2$ is not globally bounded. Nevertheless, for any initial state $x(t_0) = x_0$, the equation has the unique solution

$$x(t) = \text{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

which is well defined for all $t \geq t_0$. △

In view of the conservative nature of the global Lipschitz condition, it would be useful to have a global existence and uniqueness theorem that requires the function f to be only locally Lipschitz. The next theorem achieves that at the expense of having to know more about the solution of the system.

Theorem 3.3 *Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$. ◇

Proof: Recall the discussion on extending solutions, preceding Example 3.3. By Theorem 3.1, there is a unique local solution over $[t_0, t_0 + \delta]$. Let $[t_0, T)$ be its maximal interval of existence. We want to show that $T = \infty$. Recall (Exercise 3.26) the fact that if T is finite, then the solution must leave any compact subset of D. Since the solution never leaves the compact set W, we conclude that $T = \infty$. □

The trick in applying Theorem 3.3 is in checking the assumption that every solution lies in a compact set without actually solving the differential equation. We will see in Chapter 4 that Lyapunov's method for studying stability is very valuable in that regard. For now, let us illustrate the application of the theorem by a simple example.

Example 3.6 Consider again the system

$$\dot{x} = -x^3 = f(x)$$

of Example 3.5. The function $f(x)$ is locally Lipschitz on R. If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$. Thus, without calculating the solution, we conclude by Theorem 3.3 that the equation has a unique solution for all $t \geq 0$. △

3.2 Continuous Dependence on Initial Conditions and Parameters

For the solution of the state equation (3.1) to be of any interest, it must depend continuously on the initial state x_0, the initial time t_0, and the right-hand side function $f(t, x)$. Continuous dependence on the initial time t_0 is obvious from the integral expression

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds$$

Therefore, we leave it as an exercise (Exercise 3.28) and concentrate our attention on continuous dependence on the initial state x_0 and the function f. Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$. The solution depends continuously on y_0 if solutions starting at nearby points are defined on the same time interval and remain close to each other in that interval. This statement can be made precise with the ε–δ argument: Given $\varepsilon > 0$, there is $\delta > 0$ such that for all z_0 in the ball $\{x \in R^n \mid \|x - y_0\| < \delta\}$, the equation $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $\|z(t) - y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$. Continuous dependence on the right-hand side function f is defined similarly, but to state the definition precisely, we need a mathematical representation of the perturbation of f. One possible representation is to replace f by a sequence of functions f_m, which converge uniformly to f as $m \to \infty$. For each function f_m, the solution of $\dot{x} = f_m(t, x)$ with $x(t_0) = x_0$ is denoted by $x_m(t)$. The solution is said to depend continuously on the right-hand side function if $x_m(t) \to x(t)$ as $m \to \infty$. This approach is a little bit involved, and will not be pursued here.[6] A more restrictive, but simpler, mathematical representation is to assume that f depends continuously on a set of constant parameters; that is, $f = f(t, x, \lambda)$, where $\lambda \in R^p$. The constant parameters could represent physical parameters of the system, and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging. Let $x(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda_0) = x_0$. The solution is said to depend continuously on λ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all λ in the ball $\{\lambda \in R^p \mid \|\lambda - \lambda_0\| < \delta\}$, the equation $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda) = x_0$, and satisfies $\|x(t, \lambda) - x(t, \lambda_0)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

Continuous dependence on initial states and continuous dependence on parameters can be studied simultaneously. We start with a simple result that bypasses the issue of existence and uniqueness and concentrates on the closeness of solutions.

Theorem 3.4 *Let $f(t, x)$ be piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with a Lipschitz constant L, where $W \subset R^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of*

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

and

$$\dot{z} = f(t, z) + g(t, z), \quad z(t_0) = z_0$$

such that $y(t),\ z(t) \in W$ for all $t \in [t_0, t_1]$. Suppose that

$$\|g(t, x)\| \le \mu, \quad \forall\ (t, x) \in [t_0, t_1] \times W$$

for some $\mu > 0$. Then,

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\}$$

$\forall\ t \in [t_0, t_1]$. ◇

Proof: The solutions $y(t)$ and $z(t)$ are given by

$$\begin{aligned} y(t) &= y_0 + \int_{t_0}^{t} f(s, y(s))\ ds \\ z(t) &= z_0 + \int_{t_0}^{t} [f(s, z(s)) + g(s, z(s))]\ ds \end{aligned}$$

[6]See [43, Section 1.3], [75, Section 1.3], or [135, Section 2.5] for results on continuous dependence on parameters using this approach.

Subtracting the two equations and taking norms yield

$$\begin{aligned}\|y(t) - z(t)\| &\leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(s, y(s)) - f(s, z(s))\|\, ds \\ &\quad + \int_{t_0}^{t} \|g(s, z(s))\|\, ds \\ &\leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L\|y(s) - z(s)\|\, ds\end{aligned}$$

where $\gamma = \|y_0 - z_0\|$. Application of the Gronwall–Bellman inequality (Lemma A.1) to the function $\|y(t) - z(t)\|$ results in

$$\|y(t) - z(t)\| \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L[\gamma + \mu(s - t_0)] \exp[L(t - s)]\, ds$$

Integrating the right-hand side by parts, we obtain

$$\begin{aligned}\|y(t) - z(t)\| &\leq \gamma + \mu(t - t_0) - \gamma - \mu(t - t_0) + \gamma \exp[L(t - t_0)] \\ &\quad + \int_{t_0}^{t} \mu \exp[L(t - s)]\, ds \\ &= \gamma \exp[L(t - t_0)] + \frac{\mu}{L}\{\exp[L(t - t_0)] - 1\}\end{aligned}$$

which completes the proof of the theorem. □

With Theorem 3.4 in hand, we can prove the next theorem on the continuity of solutions in terms of initial states and parameters.

Theorem 3.5 *Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0, t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$, where $D \subset R^n$ is an open connected set. Let $y(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in D$. Suppose $y(t, \lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$. Then, given $\varepsilon > 0$, there is $\delta > 0$ such that if*

$$\|z_0 - y_0\| < \delta \quad \text{and} \quad \|\lambda - \lambda_0\| < \delta$$

then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$, and $z(t, \lambda)$ satisfies

$$\|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon, \quad \forall\, t \in [t_0, t_1]$$

◇

Proof: By continuity of $y(t, \lambda_0)$ in t and the compactness of $[t_0, t_1]$, we know that $y(t, \lambda_0)$ is bounded on $[t_0, t_1]$. Define a "tube" U around the solution $y(t, \lambda_0)$ (see Figure 3.1) by

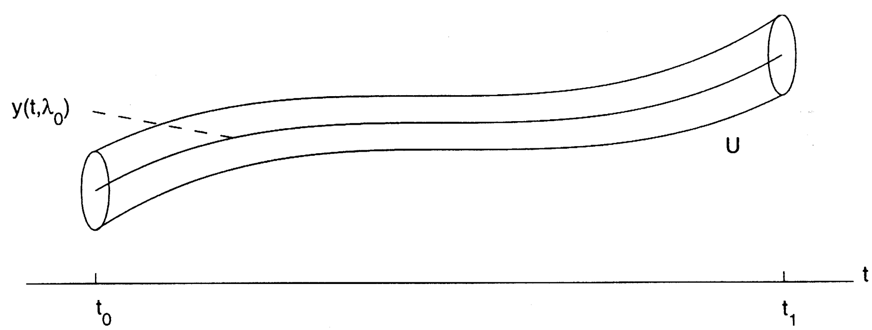

Figure 3.1: A tube constructed around the solution $y(t, \lambda_0)$.

$$U = \{(t, x) \in [t_0, t_1] \times R^n \mid \|x - y(t, \lambda_0)\| \leq \varepsilon\}$$

Suppose that $U \subset [t_0, t_1] \times D$; if not, replace ε by $\varepsilon_1 < \varepsilon$ that is small enough to ensure that $U \subset [t_0, t_1] \times D$ and continue the proof with ε_1. The set U is compact; hence, $f(t, x, \lambda)$ is Lipschitz in x on U with a Lipschitz constant, say, L. By continuity of f in λ, for any $\alpha > 0$, there is $\beta > 0$ (with $\beta < c$) such that

$$\|f(t, x, \lambda) - f(t, x, \lambda_0)\| < \alpha, \quad \forall\ (t, x) \in U, \ \forall\ \|\lambda - \lambda_0\| < \beta$$

Take $\alpha < \varepsilon$ and $\|z_0 - y_0\| < \alpha$. By the local existence and uniqueness theorem, there is a unique solution $z(t, \lambda)$ on some time interval $[t_0, t_0 + \Delta]$. The solution starts inside the tube U, and as long as it remains in the tube, it can be extended. We will show that, by choosing a small enough α, the solution remains in U for all $t \in [t_0, t_1]$. In particular, we let τ be the first time the solution leaves the tube and show that we can make $\tau > t_1$. On the time interval $[t_0, \tau]$, the conditions of Theorem 3.4 are satisfied with $\mu = \alpha$. Hence,

$$\begin{aligned} \|z(t, \lambda) - y(t, \lambda_0)\| &< \alpha \exp[L(t - t_0)] + \frac{\alpha}{L}\{\exp[L(t - t_0)] - 1\} \\ &< \alpha \left(1 + \frac{1}{L}\right) \exp[L(t - t_0)] \end{aligned}$$

Choosing $\alpha \leq \varepsilon L \exp[-L(t_1 - t_0)]/(1 + L)$ ensures that the solution $z(t, \lambda)$ cannot leave the tube during the interval $[t_0, t_1]$. Therefore, $z(t, \lambda)$ is defined on $[t_0, t_1]$ and satisfies $\|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon$. Taking $\delta = \min\{\alpha, \beta\}$ completes the proof of the theorem. □

3.3 Differentiability of Solutions and Sensitivity Equations

Suppose that $f(t, x, \lambda)$ is continuous in (t, x, λ) and has continuous first partial derivatives with respect to x and λ for all $(t, x, \lambda) \in [t_0, t_1] \times R^n \times R^p$. Let λ_0 be a nominal value of λ, and suppose that the nominal state equation

$$\dot{x} = f(t, x, \lambda_0), \quad \text{with } x(t_0) = x_0$$

has a unique solution $x(t, \lambda_0)$ over $[t_0, t_1]$. From Theorem 3.5, we know that for all λ sufficiently close to λ_0, that is, $\|\lambda - \lambda_0\|$ sufficiently small, the state equation

$$\dot{x} = f(t, x, \lambda), \quad \text{with } x(t_0) = x_0$$

has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$ that is close to the nominal solution $x(t, \lambda_0)$. The continuous differentiability of f with respect to x and λ implies the additional property that the solution $x(t, \lambda)$ is differentiable with respect to λ near λ_0. To see this, write

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda)\, ds$$

Taking partial derivatives with respect to λ yields

$$x_\lambda(t, \lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda)\, x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds$$

where $x_\lambda(t, \lambda) = [\partial x(t, \lambda)/\partial \lambda]$ and $[\partial x_0/\partial \lambda] = 0$, since x_0 is independent of λ. Differentiating with respect to t, it can be seen that $x_\lambda(t, \lambda)$ satisfies the differential equation

$$\frac{\partial}{\partial t} x_\lambda(t, \lambda) = A(t, \lambda) x_\lambda(t, \lambda) + B(t, \lambda), \quad x_\lambda(t_0, \lambda) = 0 \tag{3.4}$$

where

$$A(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right|_{x = x(t, \lambda)}, \quad B(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right|_{x = x(t, \lambda)}$$

For λ sufficiently close to λ_0, the matrices $A(t, \lambda)$ and $B(t, \lambda)$ are defined on $[t_0, t_1]$. Hence, $x_\lambda(t, \lambda)$ is defined on the same interval. At $\lambda = \lambda_0$, the right-hand side of (3.4) depends only on the nominal solution $x(t, \lambda_0)$. Let $S(t) = x_\lambda(t, \lambda_0)$; then $S(t)$ is the unique solution of the equation

$$\dot{S}(t) = A(t, \lambda_0) S(t) + B(t, \lambda_0), \quad S(t_0) = 0 \tag{3.5}$$

The function $S(t)$ is called the *sensitivity function*, and (3.5) is called the *sensitivity equation*. Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions. They can also be used to approximate the solution when λ is sufficiently close to its nominal value λ_0. For small $\|\lambda - \lambda_0\|$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$ to obtain

$$x(t, \lambda) = x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + \text{ higher-order terms}$$

Neglecting the higher-order terms, the solution $x(t, \lambda)$ can be approximated by

$$x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0) \tag{3.6}$$

We will not justify this approximation here. It will be justified in Chapter 10 when we study the perturbation theory. The significance of (3.6) is in the fact that knowledge of the nominal solution and the sensitivity function suffices to approximate the solution for all values of λ in a (small) ball centered at λ_0.

The procedure for calculating the sensitivity function $S(t)$ is summarized by the following steps:

- Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$.
- Evaluate the Jacobian matrices

$$A(t, \lambda_0) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right|_{x = x(t, \lambda_0), \lambda = \lambda_0}, \quad B(t, \lambda_0) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right|_{x = x(t, \lambda_0), \lambda = \lambda_0}$$

- Solve the sensitivity equation (3.5) for $S(t)$.

In this procedure, we need to solve the nonlinear nominal state equation and the linear time-varying sensitivity equation. Except for some trivial cases, we will be forced to solve these equations numerically. An alternative approach for calculating $S(t)$ is to solve for the nominal solution and the sensitivity function simultaneously. This can be done by appending the variational equation (3.4) with the original state equation, then setting $\lambda = \lambda_0$ to obtain the $(n + np)$ augmented equation

$$\begin{aligned} \dot{x} &= f(t, x, \lambda_0), & x(t_0) &= x_0 \\ \dot{S} &= \left[\frac{\partial f(t,x,\lambda)}{\partial x}\right]_{\lambda=\lambda_0} S + \left[\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_0}, & S(t_0) &= 0 \end{aligned} \tag{3.7}$$

which is solved numerically. Notice that if the original state equation is autonomous, that is, $f(t, x, \lambda) = f(x, \lambda)$, then the augmented equation (3.7) will be autonomous as well. We illustrate the latter procedure by the next example.

Example 3.7 Consider the phase-locked-loop model

$$\begin{aligned} \dot{x}_1 &= x_2 &&= f_1(x_1, x_2) \\ \dot{x}_2 &= -c \sin x_1 - (a + b \cos x_1) x_2 &&= f_2(x_1, x_2) \end{aligned}$$

and suppose the parameters a, b, and c have the nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$. The nominal system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \end{aligned}$$

The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \lambda]$ are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c\cos x_1 + bx_2 \sin x_1 & -(a + b\cos x_1) \end{bmatrix}$$

$$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial b} & \dfrac{\partial f}{\partial c} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2\cos x_1 & -\sin x_1 \end{bmatrix}$$

Evaluate these Jacobian matrices at the nominal parameters $a = 1$, $b = 0$, and $c = 1$ to obtain

$$\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2\cos x_1 & -\sin x_1 \end{bmatrix}$$

Let

$$S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix}\right|_{\text{nominal}}$$

Then (3.7) is given by

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(0) &= x_{10} \\
\dot{x}_2 &= -\sin x_1 - x_2, & x_2(0) &= x_{20} \\
\dot{x}_3 &= x_4, & x_3(0) &= 0 \\
\dot{x}_4 &= -x_3\cos x_1 - x_4 - x_2, & x_4(0) &= 0 \\
\dot{x}_5 &= x_6, & x_5(0) &= 0 \\
\dot{x}_6 &= -x_5\cos x_1 - x_6 - x_2\cos x_1, & x_6(0) &= 0 \\
\dot{x}_7 &= x_8, & x_7(0) &= 0 \\
\dot{x}_8 &= -x_7\cos x_1 - x_8 - \sin x_1, & x_8(0) &= 0
\end{aligned}$$

The solution of this equation was computed for the initial state $x_{10} = x_{20} = 1$. Figure 3.2(a) shows x_3, x_5, and x_7, which are the sensitivities of x_1 with respect to a, b, and c, respectively. Figure 3.2(b) shows the corresponding quantities for x_2. Inspection of these figures shows that the solution is more sensitive to variations in the parameter c than to variations in the parameters a and b. This pattern is consistent when we solve for other initial states. △

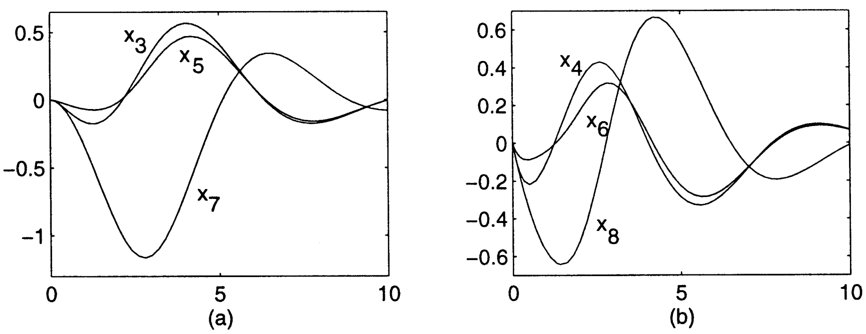

Figure 3.2: Sensitivity function for Example 3.7.

3.4 Comparison Principle

Quite often when we study the state equation $\dot{x} = f(t, x)$ we need to compute bounds on the solution $x(t)$ without computing the solution itself. The Gronwall–Bellman inequality (Lemma A.1) is one tool that can be used toward that goal. Another tool is the comparison lemma. It applies to a situation where the derivative of a scalar differentiable function $v(t)$ satisfies inequality of the form $\dot{v}(t) \leq f(t, v(t))$ for all t in a certain time interval. Such inequality is called a *differential inequality* and a function $v(t)$ satisfying the inequality is called a solution of the differential inequality. The comparison lemma compares the solution of the differential inequality $\dot{v}(t) \leq f(t, v(t))$ with the solution of the differential equation $\dot{u} = f(t, u)$. The lemma applies even when $v(t)$ is not differentiable, but has an upper right-hand derivative $D^+v(t)$, which satisfies a differential inequality. The upper right-hand derivative $D^+v(t)$ is defined in Appendix C.2. For our purposes, it is enough to know two facts:

- If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$.
- If
$$\frac{1}{h}|v(t+h) - v(t)| \leq g(t, h), \quad \forall\, h \in (0, b]$$
and
$$\lim_{h \to 0^+} g(t, h) = g_0(t)$$
then $D^+v(t) \leq g_0(t)$.

The limit $h \to 0^+$ means that h approaches zero from above.

Lemma 3.4 (Comparison Lemma) *Consider the scalar differential equation*

$$\dot{u} = f(t, u), \quad u(t_0) = u_0$$

where $f(t,u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in J \subset R$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$. ◇

Proof: See Appendix C.2.

Example 3.8 The scalar differential equation

$$\dot{x} = f(x) = -(1 + x^2)x, \quad x(0) = a$$

has a unique solution on $[0, t_1)$, for some $t_1 > 0$, because $f(x)$ is locally Lipschitz. Let $v(t) = x^2(t)$. The function $v(t)$ is differentiable and its derivative is given by

$$\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t)$$

Hence, $v(t)$ satisfies the differential inequality

$$\dot{v}(t) \leq -2v(t), \quad v(0) = a^2$$

Let $u(t)$ be the solution of the differential equation

$$\dot{u} = -2u, \quad u(0) = a^2 \quad \Rightarrow \quad u(t) = a^2 e^{-2t}$$

Then, by the comparison lemma, the solution $x(t)$ is defined for all $t \geq 0$ and satisfies

$$|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \quad \forall\, t \geq 0$$

△

Example 3.9 The scalar differential equation

$$\dot{x} = f(t, x) = -(1 + x^2)x + e^t, \quad x(0) = a$$

has a unique solution on $[0, t_1)$ for some $t_1 > 0$, because $f(t, x)$ is locally Lipschitz in x. We want to find an upper bound on $|x(t)|$ similar to the one we obtained in the previous example. Let us start with $v(t) = x^2(t)$ as in Example 3.8. The derivative of v is given by

$$\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) + 2x(t)e^t \leq -2v(t) + 2\sqrt{v(t)}e^t$$

We can apply the comparison lemma to this differential inequality, but the resulting differential equation will not be easy to solve. Instead, we consider a different choice of $v(t)$. Let $v(t) = |x(t)|$. For $x(t) \neq 0$, the function $v(t)$ is differentiable and its derivative is given by

$$\dot{v}(t) = \frac{d}{dt}\sqrt{x^2(t)} = \frac{x(t)\dot{x}(t)}{|x(t)|} = -|x(t)|[1 + x^2(t)] + \frac{x(t)}{|x(t)|}e^t$$

Since $1 + x^2(t) \geq 1$, we have $-|x(t)|[1 + x^2(t)] \leq -|x(t)|$ and $\dot{v}(t) \leq -v(t) + e^t$. On the other hand, when $x(t) = 0$, we have

$$\begin{aligned} \frac{|v(t+h) - v(t)|}{h} &= \frac{|x(t+h)|}{h} = \frac{1}{h}\left|\int_t^{t+h} f(\tau, x(\tau))\ d\tau\right| \\ &= \left|f(t,0) + \frac{1}{h}\int_t^{t+h} [f(\tau, x(\tau)) - f(t, x(t))]\ d\tau\right| \\ &\leq |f(t,0)| + \frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\ d\tau \end{aligned}$$

Since $f(t, x(t))$ is a continuous function of t, given any $\varepsilon > 0$ there is $\delta > 0$ such that for all $|\tau - t| < \delta$, $|f(\tau, x(\tau)) - f(t, x(t))| < \varepsilon$. Hence, for all $h < \delta$,

$$\frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\ d\tau < \varepsilon$$

which shows that

$$\lim_{h \to 0^+} \frac{1}{h}\int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|\ d\tau = 0$$

Thus, $D^+v(t) \leq |f(t,0)| = e^t$ whenever $x(t) = 0$. Hence, for all $t \in [0, t_1)$, we have

$$D^+v(t) \leq -v(t) + e^t, \quad v(0) = |a|$$

Letting $u(t)$ be the solution of the linear differential equation

$$\dot{u} = -u + e^t, \quad u(0) = |a|$$

we conclude by the comparison lemma that

$$v(t) \leq u(t) = e^{-t}|a| + \tfrac{1}{2}\left[e^t - e^{-t}\right], \quad \forall\, t \in [0, t_1)$$

The upper bound on $v(t)$ is finite for every finite t_1 and approaches infinity only as $t_1 \to \infty$. Therefore, the solution $x(t)$ is defined for all $t \geq 0$ and satisfies

$$|x(t)| \leq e^{-t}|a| + \tfrac{1}{2}\left[e^t - e^{-t}\right], \quad \forall\, t \geq 0$$

△

3.5 Exercises

3.1 For each of the functions $f(x)$ given next, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

(1) $f(x) = x^2 + |x|$.
(2) $f(x) = x + \text{sgn}(x)$.
(3) $f(x) = \sin(x)\,\text{sgn}(x)$.
(4) $f(x) = -x + a\sin(x)$.
(5) $f(x) = -x + 2|x|$.
(6) $f(x) = \tan(x)$.

(7) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$.

(8) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$.

3.2 Let $D_r = \{x \in R^n \mid \|x\| < r\}$. For each of the following systems, represented as $\dot{x} = f(t,x)$, find whether (a) f is locally Lipschitz in x on D_r, for sufficiently small r; (b) f is locally Lipschitz in x on D_r, for any finite $r > 0$; (c) f is globally Lipschitz in x:

(1) The pendulum equation with friction and constant input torque (Section 1.2.1).

(2) The tunnel-diode circuit (Example 2.1).

(3) The mass–spring equation with linear spring, linear viscous damping, Coulomb friction, and zero external force (Section 1.2.3).

(4) The Van der Pol oscillator (Example 2.6).

(5) The closed-loop equation of a third-order adaptive control system (Section 1.2.5).

(6) The system $\dot{x} = Ax - B\psi(Cx)$, where A, B, and C are $n \times n$, $n \times 1$, and $1 \times n$ matrices, respectively, and $\psi(\cdot)$ is the dead-zone nonlinearity of Figure 1.10(c).

3.3 Show that if $f_1 : R \to R$ and $f_2 : R \to R$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$ and $f_2 \circ f_1$ are locally Lipschitz.

3.4 Let $f : R^n \to R^n$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{\|Kx\|} Kx, & \text{if } g(x)\|Kx\| \geq \mu > 0 \\[2ex] \dfrac{g(x)}{\mu} Kx, & \text{if } g(x)\|Kx\| < \mu \end{cases}$$

where $g : R^n \to R$ is locally Lipschitz and nonnegative, and K is a constant matrix. Show that $f(x)$ is Lipschitz on any compact subset of R^n.

3.5 Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different p-norms on R^n. Show that $f : R^n \to R^m$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

3.6 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \leq k_1 + k_2\|x\|, \quad \forall\ (t,x) \in [t_0,\infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \leq \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}$$

for all $t \geq t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

3.7 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)} g(x)$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

3.8 Show that the state equation

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b \end{aligned}$$

has a unique solution defined for all $t \geq 0$.

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

3.10 Derive the sensitivity equations for the tunnel-diode circuit of Example 2.1 as L and C vary from their nominal values.

3.11 Derive the sensitivity equations for the Van der Pol oscillator of Example 2.6 as ε varies from its nominal value. Use the state equation in the x-coordinates.

3.12 Repeat the previous exercise by using the state equation in the z-coordinates.

3.13 Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

3.14 Consider the system

$$\dot{x}_1 = -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)$$
$$\dot{x}_2 = -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)$$

where λ and τ are positive constants.

(a) Derive the sensitivity equations as λ and τ vary from their nominal values λ_0 and τ_0.

(b) Show that $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \le -\frac{1}{\tau}r + 2\sqrt{2}$$

(c) Using the comparison lemma, show that the solution of the state equation satisfies the inequality

$$\|x(t)\|_2 \le e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau})$$

3.15 Using the comparison lemma, show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}, \qquad \dot{x}_2 = -x_2 + \frac{2x_1}{1+x_1^2}$$

satisfies the inequality

$$\|x(t)\|_2 \le e^{-t}\|x(0)\|_2 + \sqrt{2}\,(1 - e^{-t})$$

3.16 Using the comparison lemma, find an upper bound on the solution of the scalar equation

$$\dot{x} = -x + \frac{\sin t}{1+x^2}, \qquad x(0) = 2$$

3.17 Consider the initial-value problem (3.1) and let $D \subset R^n$ be a domain that contains $x = 0$. Suppose $x(t)$, the solution of (3.1), belongs to D for all $t \ge t_0$ and $\|f(t,x)\|_2 \le L\|x\|_2$ on $[t_0, \infty) \times D$. Show that

(a)

$$\left| \frac{d}{dt}\left[x^T(t)x(t)\right] \right| \le 2L\|x(t)\|_2^2$$

(b)

$$\|x_0\|_2 \exp[-L(t-t_0)] \le \|x(t)\|_2 \le \|x_0\|_2 \exp[L(t-t_0)]$$

3.18 Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \leq k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3] \, d\tau$$

where k_1, k_2, and k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall–Bellman inequality, show that

$$y(t) \leq k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

3.19 Let $f : R^n \to R^n$ be locally Lipschitz in a domain $D \subset R^n$. Let $S \subset D$ be a compact set. Show that there is a positive constant L such that for all $x, y \in S$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Hint: The set S can be covered by a finite number of neighborhoods; that is,

$$S \subset N(a_1, r_1) \cup N(a_2, r_2) \cup \cdots \cup N(a_k, r_k)$$

Consider the following two cases separately:

- $x, y \in S \cap N(a_i, r_i)$ for some i.
- $x, y \notin S \cap N(a_i, r_i)$ for any i; in this case, $\|x - y\| \geq \min_i r_i$.

In the second case, use the fact that $f(x)$ is uniformly bounded on S.

3.20 Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

3.21 For any $x \in R^n - \{0\}$ and any $p \in [1, \infty)$, define $y \in R^n$ by

$$y_i = \frac{x_i^{p-1}}{\|x\|_p^{p-1}} \operatorname{sign}(x_i^p)$$

Show that $y^T x = \|x\|_p$ and $\|y\|_q = 1$, where $q \in (1, \infty]$ is determined from $1/p + 1/q = 1$. For $p = \infty$, find a vector y such that $y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$.

3.22 Prove Lemma 3.3.

3.23 Let $f(x)$ be a continuously differentiable function that maps a convex domain $D \subset R^n$ into R^n. Suppose D contains the origin $x = 0$ and $f(0) = 0$. Show that

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) \, d\sigma \, x, \quad \forall \, x \in D$$

Hint: Set $g(\sigma) = f(\sigma x)$ for $0 \leq \sigma \leq 1$ and use the fact that $g(1) - g(0) = \int_0^1 g'(\sigma) \, d\sigma$.

3.24 Let $V : R\times R^n \to R$ be continuously differentiable. Suppose that $V(t,0) = 0$ for all $t \ge 0$ and

$$V(t,x) \ge c_1\|x\|^2; \quad \left\|\frac{\partial V}{\partial x}(t,x)\right\| \le c_4\|x\|, \quad \forall\ (t,x) \in [0,\infty)\times D$$

where c_1 and c_4 are positive constants and $D \subset R^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \le \frac{1}{2}c_4\|x\|^2$ for all $x \in D$.
Hint: Use the representation $V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t,\sigma x)\ d\sigma\ x$.

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \le c_4$.

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$|W(t,x_2) - W(t,x_1)| \le \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\|, \quad \forall\ t \ge 0,\ \forall\ x_1, x_2 \in D$$

3.25 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0,t_1]\times D$, for some domain $D \subset R^n$. Let W be a compact subset of D. Let $x(t)$ be the solution of $\dot{x} = f(t,x)$ starting at $x(t_0) = x_0 \in W$. Suppose that $x(t)$ is defined and $x(t) \in W$ for all $t \in [t_0,T)$, $T < t_1$.

(a) Show that $x(t)$ is uniformly continuous on $[t_0,T)$.

(b) Show that $x(T)$ is defined and belongs to W and $x(t)$ is a solution on $[t_0,T]$.

(c) Show that there is $\delta > 0$ such that the solution can be extended to $[t_0, T+\delta]$.

3.26 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0,t_1]\times D$, for some domain $D \subset R^n$. Let $y(t)$ be a solution of (3.1) on a maximal open interval $[t_0,T) \subset [t_0,t_1]$ with $T < \infty$. Let W be any compact subset of D. Show that there is some $t \in [t_0,T)$ with $y(t) \notin W$.
Hint: Use the previous exercise.

3.27 ([43]) Let $x_1 : R \to R^n$ and $x_2 : R \to R^n$ be differentiable functions such that

$$\|x_1(a) - x_2(a)\| \le \gamma, \quad \|\dot{x}_i(t) - f((t,x_i(t))\| \le \mu_i, \text{ for } i = 1,2$$

for $a \le t \le b$. If f satisfies the Lipschitz condition (3.2), show that

$$\|x_1(t) - x_2(t)\| \le \gamma e^{L(t-a)} + (\mu_1+\mu_2)\left[\frac{e^{L(t-a)}-1}{L}\right], \quad \text{for } a \le t \le b$$

3.28 Show, under the assumptions of Theorem 3.5, that the solution of (3.1) depends continuously on the initial time t_0.

3.29 Let $f(t, x)$ and its partial derivatives with respect to x be continuous in (t, x) for all $(t, x) \in [t_0, t_1] \times R^n$. Let $x(t, \eta)$ be the solution of (3.1) that starts at $x(t_0) = \eta$ and suppose $x(t, \eta)$ is defined on $[t_0, t_1]$. Show that $x(t, \eta)$ is continuously differentiable with respect to η and find the variational equation satisfied by $[\partial x/\partial \eta]$. Hint: Put $y = x - \eta$ to transform (3.1) into

$$\dot{y} = f(t, y + \eta), \quad y(t_0) = 0$$

with η as a parameter.

3.30 Let $f(t, x)$ and its partial derivative with respect to x be continuous in (t, x) for all $(t, x) \in R \times R^n$. Let $x(t, a, \eta)$ be the solution of (3.1) that starts at $x(a) = \eta$ and suppose that $x(t, a, \eta)$ is defined on $[a, t_1]$. Show that $x(t, a, \eta)$ is continuously differentiable with respect to a and η and let $x_a(t)$ and $x_\eta(t)$ denote $[\partial x/\partial a]$ and $[\partial x/\partial \eta]$, respectively. Show that $x_a(t)$ and $x_\eta(t)$ satisfy the identity

$$x_a(t) + x_\eta(t) f(a, \eta) \equiv 0, \quad \forall\ t \in [a, t_1]$$

3.31 ([43]) Let $f : R \times R \to R$ be a continuous function. Suppose that $f(t, x)$ is locally Lipschitz and nondecreasing in x for each fixed value of t. Let $x(t)$ be a solution of $\dot{x} = f(t, x)$ on an interval $[a, b]$. If a continuous function $y(t)$ satisfies the integral inequality

$$y(t) \le x(a) + \int_a^t f(s, y(s))\, ds$$

for $a \le t \le b$, show that $y(t) \le x(t)$ throughout this interval.