Consider the trivial version of system under twisting controller

$$\ddot{x} = -b_1 \operatorname{sign} x - b_2 \operatorname{sign} \dot{x}, \quad b_1 > b_2 > 0. \tag{1}$$

We can consider the physical meaning of (1) like the mechanical system with unit mass, Coulomb friction $-b_2 \operatorname{sign} \dot{x}$, and position control $-b_1 \operatorname{sign} x$. It is clear that to compensate the friction term we should have $b_1 > b_2 > 0$.

Proposition 1 *Solutions of (1) converge to the origin for a finite time.*

Proof. The vector field of equation is drown in the fig. 1. It is clear that

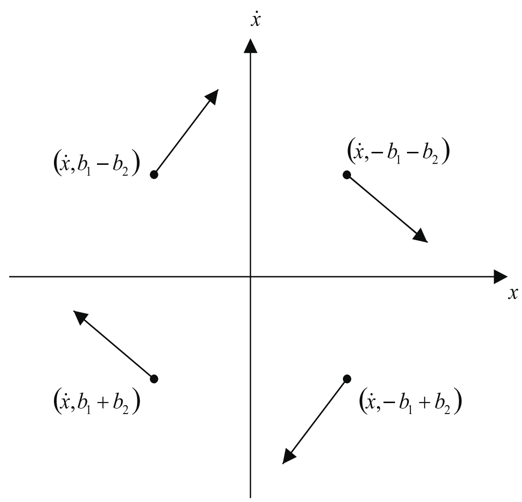

Figure 1: Vector field for trivial system under twisting controller

solutions of system (1) cross the axis $x = 0$ for the finite time As always with the Filippov definitions the values taken on a set of the measure 0 do not matter. Assume now for simplicity that the initial values are $x = 0$, $\dot{x} = \dot{x}_0 > 0$ at $t = 0$. Thus, the trajectory enters the half-plane $x > 0$. Let $\dot{x}_0 x_1 \dot{x}_1$ (Fig.2) be the trajectory of the equation (1).The trajectories of equation (1) is defined by

$$\frac{d(\dot{x})}{dx} = \frac{-b_1 - b_2}{\dot{x}} \text{ with } x > 0, \dot{x} > 0$$

$$\frac{d(\dot{x})}{dx} = \frac{-b_1 + b_2}{\dot{x}} \text{ with } x > 0, \dot{x} < 0$$

Simple calculation shows that with $x > 0$ the solution of (1) starting from the point $(x_1, 0)$ is determined by the equalities

$$\begin{array}{ll} * \ x = x_1 - \frac{\dot{x}^2}{2(b_1 + b_2)} & \text{with } \dot{x} > 0, \\ \ \ x = x_1 - \frac{\dot{x}^2}{2(b_1 - b_2)} & \text{with } \dot{x} \leq 0, \end{array} \tag{2}$$

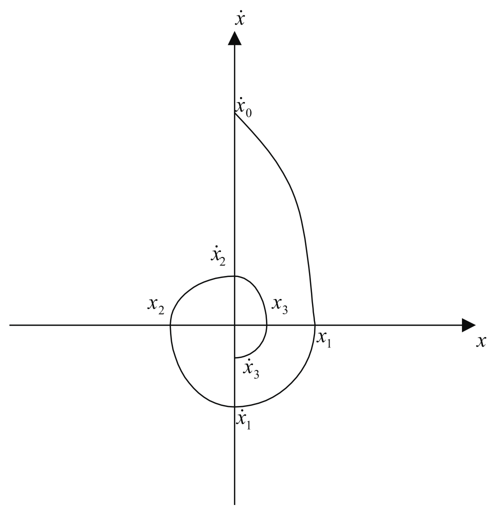

Figure 2: Twisting trajectory

For the points $x(0) = 0, \dot{x}(0) = \dot{x}_0$ and $x(0) = 0, \dot{x}(0) = \dot{x}_M$ one has $2(b_1 + b_2)x_1 = \dot{x}_0^2$, and $2(b_1 - b_2)x_1 = \dot{x}_1^2$ correspondingly. Consequently

$$|\dot{x}_1|/\dot{x}_0 = \sqrt{\frac{b_1 - b_2}{b_1 + b_2}} = q < 1.$$

Extending the trajectory into the halfplane $x < 0$ after a similar reasoning

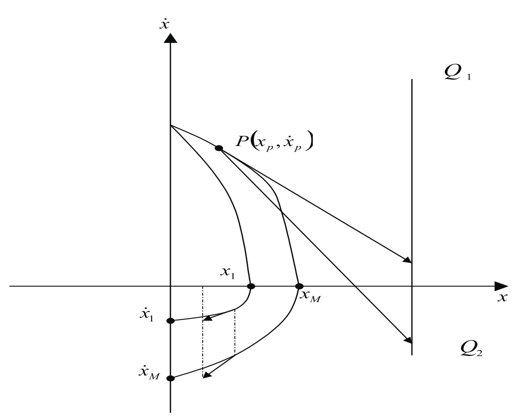

Figure 3: Majorant curve of the twisting controller

achieve that the successive crossings of the axis $x = 0$ satisfy the inequality $|\dot{x}_{i+1}|/|\dot{x}_i| = q < 1$ (Fig. 2). Therefore, the algorithm obviously converges.

The convergence time is to be estimated now. The real trajectory consists of infinite number of segments belonging to the half-planes $x \geq 0$ and $x \leq 0$.

Let us estimate the time t_1^+ on the part of the system (1) trajectory $\dot{x}_0 x_1$.

Integrating the equation (1) one has $\dot{x}(t) = -(b_1 + b_2)t + \dot{x}_0$. Thus $\dot{x}(t_1^+) = 0 \;\rightarrow\; t_1^+ = \dot{x}_0/(b_1 + b_2)$. Now for t_1^- corresponding to $x_1 \dot{x}_1$ one can compute $x(t) = -\frac{(b_1 - b_2)t^2}{2} + x_1, \rightarrow$

$$x(t_1^-) = 0 \;\rightarrow\; t_1^- = \sqrt{\frac{2x_1}{b_1 - b_2}} = \sqrt{\frac{1}{(b_1 - b_2)(b_1 + b_2)}}\dot{x}_0.$$

This means that the time interval

$$t_1 = t_1^+ + t_1^- = \eta \dot{x}_0, \eta = \frac{1}{(b_1 + b_2)} + \sqrt{\frac{1}{(b_1 - b_2(b_1 + b_2)}}$$

corresponding to the trajectory $\dot{x}_0 x_1 \dot{x}_1$. Consequently the time interval

$$t_i = \eta |\dot{x}_{i-1}| = \eta q^{i-1} \dot{x}_0$$

corresponding to $\dot{x}_{i-1} x_1 \dot{x}_i$. In this case for total convergence time is the follows:

$$T = \sum_{i=1}^{\infty} t_i = \sum_{i=1}^{\infty} \eta |\dot{x}_{i_1}| = \sum_{i=1}^{\infty} \eta q^{i-1} \dot{x}_0 = \frac{\eta \dot{x}_0}{1 - q}.$$

■

Consider a simple example. Let an uncertain dynamic system be given by the following differential equation

$$\ddot{x} = a(t) + b(t)u, \tag{3}$$

$$|a(t,x)| \leq C, \quad 0 \leq K_m \leq b(t,x) \leq K_M, \tag{4}$$

$$u = -r_1 \operatorname{sign} x - r_2 \operatorname{sign} \dot{x}, \quad r_1 > r_2 > 0. \tag{5}$$

Lemma 2 *Let r_1 and r_2 satisfy the conditions*

$$K_m(r_1 + r_2) - C > K_M(r_1 - r_2) + C, \; K_m(r_1 - r_2) > C. \tag{6}$$

Then controller (5) provides for the appearance of a 2-sliding mode $x = \dot{x} = 0$ attracting the trajectories of the system (3) in finite time.

Proof. Let us define $b_1 > b_2 > 0$ as follows:

$$b_1 + b_2 = K_m(r_1 + r_2) - C, \; -b_1 + b_2 = K_M(r_1 - r_2) + C$$

The system (1) for such a choice of parameters b_1, b_2 is called the majorant system. Let us denote as $\dot{x}_0 x_1 \dot{x}_1$ and $\dot{x}_0 x_M \dot{x}_M$ (Fig. 3) the trajectory of the differential equation(3) and (1) with the initial values are $x = 0$, $\dot{x} = \dot{x}_0 > 0$

at $t = 0$. Let us call the trajectory $\dot{x}_0 x_M \dot{x}_M$ as majorant curve. Consider any point $P(x_P, \dot{x}_P)$ of this curve (Fig. 3) belongs to the quadrant $Q_1 = \{(x, \dot{x}) : x \geq 0,\ \dot{x} \geq 0\}$. The velocity of (3), (5) at this point has coordinates $(\dot{x}_P, \ddot{x}_P)$. Hence, the horizontal component of the velocity depends only on the point itself. Since the vertical component satisfies the inequalities (6), the velocity of (3), (5) always "looks" into the region bounded by the axis $x = 0$ and curve (1). This means that the trajectory of system (3), (5) intersects with the axis $\dot{x} = 0$ at the point $x_1 \leq x_M$, the and the time corresponding time intervals $t_1^+ < t_M^+$. Consider the trajectories $x_1\dot{x}_1$ and $x_M\dot{x}_M$ of systems (3), (5) and (1) correspondingly in the quadrant $Q_2 = \{(x, \dot{x}) : x \geq 0,\ \dot{x} \leq 0\}$. As it follows from the (6) in Q_2 the modulo of vertical component of the velocity vector of (3), (5) is less that for the majorant system (1) but the horizontal components are the same (see Fig. 3). This means that the trajectory of the system (3), (5) in Q_2 lies in the interior of the majorant trajectory. On the hand, the time interval to cover the horizontal segment $[0, x_1]$ is the same but majornat trajectory needs to cover the horizontal segment $[x_1, x_M]$. That is why for the time intervals t_1^-, t_M^- corresponding to the trajectory $x_1\dot{x}_1$ of system (3), (5) and $x_M\dot{x}_M$ of majorant one has $t_1^- < t_M^-$.

This means that the trajectory of system (3), (5) intersect the next time with the axis $x = 0$ at the point $\dot{x}_1$. Moreover $|\dot{x}_1| \leq |\dot{x}_M|$, and the time corresponding time intervals $t_1 < t_M$. Finite time convergence of the system (3), (5) now follows from convergence of majorant system (1) proved in the Proposition 1. ■